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Encyclopedia of the History of Arabic Science

Volume 2

Mathematics and the Physical Sciences

Edited by
Roshdi Rashed

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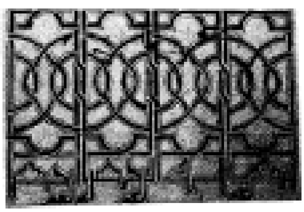

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Numeration and arithmetic

AHMAD S.SAIDAN

The earliest Arabic works of arithmetic ever written are those by al-Khwārizmī, Muḥammad ibn Mūsā (ninth century). He wrote two tracts: one, on Hindu arithmetic, has come down to us in a Latin tradition,[1] and the other, named *al-Jamʿ wa al-Tafrīq* ('Augmentation and diminution'), is mentioned in Arabic bibliographies,[2] and we have a quotation of it in an Arabic work (al-Baghdādī, *al-Takmila*). The earliest Arabic arithmetic that has come to us in full is the arithmetic of al-Uqlīdisī, Aḥmad ibn Ibrāhīm (tenth century) (al-Uqlīdisī, *al-Fuṣūl,* p. 349). It discusses a Hindu system of calculation and refers to two others, namely, finger-reckoning and the sexagesimal system. These three systems, together with the Greek *arithmetica,* which is in fact rudiments of number theory, formed the elements of arithmetic and were amalgamated and developed.

THE SCALE OF SIXTY

In Arabic books, the scale of sixty is called the method of astronomers. The sexagesimal system comprises the major arithmetical operations in the scale of sixty. It came down to the Islamic world from the ancient Babylonians through Syriac and Persian channels. We have no early work devoted to it, but we find it in all arithmetic works amalgamated with one or both of the Hindu system and finger-reckoning. Later works that stress it contain hardly anything purely arithmetical which is not involved in other works nor bearing traces of Islamic development. Scholars today find it easier than the decimal system when treating medieval astronomical calculations. It has now disappeared from use, but leaves its trace in our submultiples of the degree and hour.

FINGER-RECKONING

In Arabic works, finger-reckoning is called the arithmetic of the Rūm (i.e. the Byzantines) and the Arabs. When and how it came down to the Islamic world cannot be known for sure. But we can guess that before Islam the Arab merchants learnt to count by fingers from their neighbours. The system seems to have been spread all over the then civilized world. Some prophetic sayings involve allusion to denoting numbers by finger signs, which is characteristic of the system.

Basically the system is mental. Addition and subtraction involve no serious difficulty. Multiplication, division and ratio can be more complicated. Hence these form most of the works in this system. Too many schemes of multiplication are given; but the main ones are shortcuts that are still used. False position and double-false position, which implies the principle of linear interpolation, are used in the case of ratio and proportion.

Extraction of square roots seems to be done mainly by rough approximation.

The working is done mentally. But intermediate results need to be temporarily 'held'. This is done by bending the fingers of the two hands in distinct poses that can denote numbers from 1 up to 9999. In *The Arithmetic of al-Uqlīdisī* the different poses are given. These are what Arabic works call "*uqūd*" (knots) and therefore the whole system is called '*ḥisāb al-'Uqūd*', i.e. the arithmetic of knots (finger joints).

Numerals in this system are the letters of the Arabic alphabet in a fixed order called *jummal,* which gives another name to the system, namely jummal arithmetic. The letters of the alphabet in the Eastern order, with the numbers they denote, are given below.

ا	1	A	ح	8	H	س	60	S	ت	400	T
ب	2	B	ط	9	I	ع	70	O	ث	500	U
ج	3	C	ي	10	J	ف	80	P	خ	600	V
د	4	D	ك	20	K	ص	90	Y	ذ	700	Z
ه	5	E	ل	30	L	ق	100	Q	ض	800	W
و	6	F	م	40	M	ر	200	R	ظ	900	I′
ز	7	G	ن	50	N	ش	300	X	غ	1000	O′

Thus to denote 1111 they wrote O′QJA; for 2000 they wrote BO′; for 1000000 O′O′. Thus theoretically they could express any number in this system of numeration. But big numbers do not appear in the extant works on finger-reckoning, because they made free use of the scale of sixty, for which letters from A to N are used.

In the Western wing of the Muslim world, the jummal order of letters is different, but the difference comes beyond the letter N and thus does not affect the sixty scale.

The earliest known work on finger-reckoning is that of Abū al-Wafā' al-Būzjānī (tenth century).[3] Not much later appeared *al-Kāfī fī* al-Ḥisāb by al-Karajī.[4] These are the only two worthwhile works on this system. It started to disappear as the Hindu system spread, leaving only shortcuts of multiplication and division together with an Arabic concept of fractions.

Although finger-reckoning came down to the Arabic-speaking world from others who spoke mostly Syriac, as it appears in the works of Abū al-Wafā' and al-Karajī it looks as if made to fit exactly the Arabic language, especially when we consider the concept of fractions. In Arabic there are nine single words to denote nine distinct fractions of the unit numerator; these are $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{10}$. These are the only *kusūr* (i.e. fractions) in the system; each is *kasr* (fraction). Even $\frac{2}{3}, \frac{3}{4}$,…are each *kusūr,* plural of *kasr*. An expression like $\frac{1}{15}$ is pronounced as part of $\frac{1}{15}$, and in calculations should be expressed as $\frac{1}{3}\times\frac{1}{5}$. Fractions of denomination other than 2, 3, 5 and 7, e.g. $\frac{1}{11}$ and $\frac{3}{13}$, are considered surds (*ṣumm*, literally deaf) and should better be changed by approximation into spoken (*munṭaq*) fractions. In Abū al-Wafā's arithmetic many pages are devoted to showing how best to turn such 'parts' into fractions. The key method is to exploit the scale of sixty. Thus $\frac{7}{15} = \frac{7}{15} \times 60$ of a degree $= 28' = 20' + 8' = \frac{1}{3} + \frac{1}{5} \times \frac{2}{3}$. $\frac{2}{3}$ was the only acceptable ‘fraction’ with a numerator other than unity. The concept makes practical calculation easy. But it is naive, anti-generalizational and not mathematical.

There were several fractional systems in finger-reckoning, the foremost being the scale of sixty, i.e. the submultiples of the degree. But any scale of measurement, of length, area, volume, or size, or transaction could be used as a fractional scale. Thus where 1 dirham=24 kirats, 1 kirat stands for $\frac{1}{24}$.

These systems faded away, and the concept of the general fraction *a/b* established itself in the Islamic era, apparently with the spread of the Hindu system. But the trend to express a fraction like $\frac{1}{15}$ as $\frac{1}{3} \times \frac{1}{5}$ continued and can still be noticed among illiterate people.

THE HINDU SYSTEM

Our current schemes of numeration and calculation owe much to the Hindu system. It seems to have started much earlier then the ninth century when al-Khwārizmī wrote about it. In the seventh century AD there lived in the convent of Kenneshre on the Euphrates in Syria a learned tituler bishop called Severus Sebokht. He wrote on several topics. In a fragment of his writings dated 662 that has come down to us, he expresses his admiration of the Hindus in comparison with the Greeks by saying:

> I will omit all discussion of the science of the Hindus,…their subtle discoveries,…discoveries that are more ingenious than those of the Greeks and the Babylonians; their valuable methods of calculation; and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs.
>
> (Smith 1923: vol. 1, pp. 166–7)

The system might have started much earlier in India and might have reached Syria through trade. Yet no reference to it is spotted in Indian writings prior to al-Khwārizmī, which may imply that traders and business men invented it and used it whilst the highbrow were ignorant of it or looked down at it.

Thanks to al-Uqlīdisī who tells us much about it, there was something in it which, at first sight at least, urges one to take a second look: its working is in dust or sand. The scribe carries a board; he strews sand on it and writes the digits he wants to deal with in the sand using his fingers or a crooked rod. Rubbing out and shifting digits goes on until all the working is rubbed off and the final answer remains.

This board, the abacus that the Muslim world used, is called *takht,* a Persian word. This should not necessarily imply that the Muslims took the system from Persia; but they might have taken it through Persia; or a Persian might have been the first to introduce it. Linguistic connotations are anyhow too intriguing to be taken as decisive implications. Those who took over the system and introduced it into Arabic called it Indian, and Indian it is.

Its distinctive characteristic is that with nine digits and a zero sign it can denote any number, however big it may be, in the decimal scale, which is the scale used in everyday life. This is through the idea of place-value; the digit 1 in the units place is one unit; in the tens place it is one ten; in the hundreds place it is one hundred, and so on.

The Babylonian sexagesimal scale had two signs and the place-value order, but in the scale of sixty. The scribe had to receive the numbers in the decimal scale, transform them to the scale of sixty, calculate and find the answer and then transform his result back into the decimal scale. Although the scale of sixty was invented in Babylonia, it remained alien to Babylonian everyday life until it was replaced by the Hindu system. Nevertheless it was the best and the most mathematical and promising scale before the Indian system.

The new system made calculations easy and open to improvement and development. The Greeks took geometry to perfection. To make any further development in mathematics required new tools. These were algebra and more developed schemes of calculations and were supplied by the Arabic mathematics with the help of Hindu arithmetic.

FORMS OF THE NUMERALS

Good pictures of the forms of the numerals can be found in most works on the history of medieval mathematics. What is given here is what the author has traced in over thirty manuscripts all over the Eastern and the Western wings of the Muslim world.

1. In early works this appears as ١̄. The little horizontal bar is usually put over every number to distinguish it from the words around it. This is Indian practice. When two numbers come together, their two bars distinguish them. Thus ١̄ ١̄ differs from ١١̄. In the hands of Arabic-writing scribes, the bar was dropped as time went on, and the numeral was lengthened to become | so that it might be distinguished from the first letter of the Arabic alphabet.

2, 3 Far eastward, in Pakistan, Iran and Afghanistan, these took the forms ۲, ۳. In Iraq and Syria, they appear as ٢, ٣ or ٢, ٣. In Western Islam they look like 2, 3.

4 In the East its early form was ٤, which was gradually changed into ٤. In the West it took the form ٤. But in the hand of the scribe it may become ۴, as if it is 3 reversed.

5 In the earliest manuscripts, it looked like ٥ or B. Later it took the form B which was changed in the far East into ۵. At present, it is ۵ in Iran, and its neighbourhood. In Western Islam it was ٦.

6 In Eastern Islam, it was ٦; in the West, 6.

7, 8, 9 In Eastern Islam they were ٧, ٨, ٩; in the West, 7, 8, 9.

0 Originally it was a little circle. In the East, 5 was changed into a circle and 0 into a dot.

It is worthwhile noting that in Arabic, these forms were called *ḥurūf al-hind,* i.e. Indian letters. They were even used for secret writings (cf. al-Uqlīdisī, *al-Fuṣūl*, p. 442).

CONTENTS OF HINDU ARITHMETIC

In the earliest work on Hindu arithmetic (Vogel 1963), the following are given in order:

1 introduction of the numerals and place-value,
2 addition and subtraction,
3 duplication and mediation,
4 multiplication,
5 division,
6 fractions, in the decimal and the sexagesimal scales,
7 multiplication of sexagesimal fractions,
8 division of sexagesimal fractions,
9 addition, subtraction, duplication and mediation of fractions,
10 multiplications of common fractions,
11 division of common fractions,
12 square roots.

All this is covered in sixteen pages of about 530 lines.

For comparison we list below the chapters of the first part of ***al-Fuṣūl,*** in which al-Uqlīdisī meant to cover the essentials of Hindu arithmetic as presented in the then current texts:

1 introduction of the numerals and place value,
2 duplication and mediation,
3 addition and subtraction,
4 multiplication, whole numbers,
5 division,
6 multiplication of a fraction by a whole number,
7 addition of fractions,
8 multiplying a composite fraction by a number,
9 multiplying fractions by fractions,
10 multiplying a composite fraction by a fraction,
11 multiplying composites by composites,
12 extracting roots of square and non-square numbers,
13 extracting roots of fractions and fractional numbers,
14 general discussion of division of fractions,
15 the scale of degrees,
16 duplication and mediation of degrees,
17 addition and subtraction of degrees and minutes,
18 multiplication of degrees and minutes,
19 division of degrees and minutes,
20 square roots of degrees and minutes,
21 square roots of surds.

By composite fraction or number we mean a whole number plus a fraction, e.g. $2\frac{1}{3}$.

We cannot assume that the Latin version (see Vogel 1963) gives all the Hindu arithmetic handed over to Islam. We cannot as well assume that all part 1 of ***al-Fuṣūl*** is free from Arabic additions. The truth lies somewhere in between. Again we cannot assume that all the Hindu arithmetic handed over to Arabic was what al-Khwārizmī introduced in his book. However, in fairness to all, we can tell for sure that the decimal scale with the operations of addition, subtraction, multiplication, division and square root extraction, of whole numbers and fractions, as well as the same operations in the scale of sixty, were taken over from India. Probably the Indian lore stressed small numbers; probably it was not much sophisticated; but the general theme, order and essence were Indian. These, added to whatever arithmetical knowledge survived through the ages and was imported to the graduates of Gundīshāpūr and other centres of learning, formed the background arithmetic on which the Arabic world built its distinguished mathematical structure. Before turning to this structure, we have to look deeper into the nature of this Indian lore that invaded the Muslim mind.

NATURE OF HINDU ARITHMETIC

Here we stress again that this system invaded Islam with the dustboard, rubbing off numbers and shifting. To understand fully what this means, let us examine the scheme for multiplying, say, 9234 by 568. We show here the first method given in all texts of Hindu arithmetic.

The numbers are set up on the board in the following way

```
  9234
568
```

This implies that 9 is to be multiplied in order by 5, 6 and 8, the products must be put in the upper line, each above the digit multiplied by 9. Thus 45 is put above 5. When 9 is multiplied by 6 to yield 54, 4 is put above 6, but 5 is to be added to 45, making 50. Thus 45 is rubbed out, and 50 inserted. Now 9 is multiplied by 8, giving 72, which is to go above: 2 takes the place of 9 and 7 is added to 4, giving 11; thus 4 is rubbed out and 1 is put in its place; the other 1 is added to 0, which requires rubbing out 0 and inserting 1. The whole structure becomes

```
5112234
 568
```

Now 568 is to be shifted one place to the right, so that its units place comes under the next digit to multiply it. The array becomes

```
5112234
  568
```

denoting that 2 is to be multiplied by 5, 6 and 8 in order. After multiplying 5, 6 and 8 by 2 and adding the products to the line above we get

5225634
568

Again 568 is to be shifted so that 8 comes under 3. The operation is repeated again and again. At last we get the required product on the upper line and 568 below it. The number which was multiplied has disappeared, leaving no chance for revision of the working, not to mention dirty fingers and dusty clothes. Easy as it may be, the algorism was in need of improvement.

AN ARABIC DEVELOPMENT OF ARITHMETIC

The required improvement was one of the first Arabic achievements. The work of al-Uqlīdisī (*al-Fuṣūl*) tells partly the story of the first attempts in this line: to shift from using the dustboard to using ink and paper and showing the separate steps of the working, so that revision is possible. Although this may look to us easy, practically it was not; with slow intercommunications and conservatism of those amongst whom the dustboard was deep rooted, the change took a long time to come about. It started in the tenth century in Damascus, according to al-Uqlīdisī, without Baghdad knowing of it. In the thirteenth century, Ibn al-Bannā' (1256–1321) shows little signs of the dustboard schemes in his writings. Yet, far East in Marāgha the great Naṣīr al-Dīn al-Ṭūsī (d. 1274) wrote a full work on dustboard arithmetic (al-Tūsī, *Jawāmi' al-ḥisāb*). About half a century before him, his predecessor Sharaf al-Dīn al-Ṭūsī[5] laboured hard to solve his cubic equations arithmetically by dustboard methods. However, the dustboard was finally given up and forgotten. The result was the system of arithmetic operations that we were taught when we were young, and that the present electronic calculators have not yet been able to make us all forget.

To separate Hindu arithmetic from the dustboard is an achievement no less important than spotting the system and preferring it to finger-reckoning. But still the latter system survived longer in the Islamic concept of fractions.

COMMON (VULGAR) AND DECIMAL FRACTIONS IN THE HINDU SYSTEM

The concept of the fraction $1/b$ is Indian. But in India it was written as

$$\begin{matrix} a \\ b \end{matrix}$$

Similarly

$$a\,\frac{b}{c}$$

was written

$$\begin{matrix} a \\ b \\ c \end{matrix}$$

This is in fact what remains on the dustboard when $ac+b$ is divided by c. Thus 19÷4 leaves

$$\begin{matrix} 4 \\ 3 \\ 4 \end{matrix}$$

when the division is completed.

The Muslims learned this set-up, but applied it to their practice of exchanging fractions to groups of unit numerators. Thus they understood $\frac{3}{4}$, but preferred to express it as $\frac{1}{2}+\frac{1}{4}$. This they wrote in the Hindu form

$$\begin{matrix} 1 \\ 2 \\ 1 \\ 4 \end{matrix}$$

The latter arrangement, however, they might read as $\frac{1}{2}\,\frac{1}{4}$. Such confusion might have accelerated the tendency to shift to the general form *a/b*. The first step that we can trace was to present, say $4\frac{3}{4}$ as

$$\begin{matrix} 4 \\ \hline 3 \\ 4 \end{matrix}$$

with the bar separating the whole number from the fraction, but still $\frac{3}{4}$ was to be changed to $\frac{1}{2}+\frac{1}{4}$. Even $\frac{1}{2}$ when by itself might be written as

$$\begin{matrix} 0 \\ 1 \\ 2 \end{matrix}$$

Ibn al-Bannā', or his predecessors in the West, accepted fully the idea of the general common fraction *a/b* and set it up as

$$\frac{a}{b}$$

with numerator and denominator separated by a bar; but they wrote a quantity like $4\frac{3}{4}$ as

$$\frac{3}{4}\ 4$$

ignoring the place-value arrangement. Al-Ṭūsī, far eastward preferred the concept of *a/b,* put aside the idea of the unit numerator, but used the bar to separate the whole number only. Later professional teachers, who apparently wrote no extant works, but have left margins on works of others, show knowledge of the structure

$$a\frac{b}{c}.$$

The form *b/c* is a late European innovation.

Al-Uqlīdisī seems to be the first to write about the decimal fractions. That was in 952 AD (al-Uqlīdisī, *al-Fuṣūl,* English translation, pp. 481–8).

One of the most remarkable ideas in the arithmetic of al-Uqlīdisī is the use of decimal fractions.[6] He suggests the concept as a device, and uses a decimal sign. He insists that this sign should always be used; he writes not less than fourteen decimal fractions, but the scribe writes the sign in only two cases. He introduces the idea for dealing with submultiples of ten in the same way as submultiples of 60 are dealt with in the sexagesimal scale. He does this in the following problems:

1 To halve 19 successively, al-Uqlīdisī gets the following:

19
9.5
4.75
2.375
1.1875
0.59375

The last number he reads as 59375 of a hundred thousand. Then by duplation he regains the 19, dropping out the zeros on the right because they are of no significance.

2 To halve 13 successively he obtains 6.5, 3.25, 1.625 and 0.8125.

3 To increase 135 by one-tenth, five successive times, he multiplies by 11 and takes one-tenth of the product. Thus in the first step he calculates 135×11=1485. This he changes to 148.5. We can now say that he is multiplying by 1.1. The next step is 148.5×1.1=163.35. Here he multiplies 148 by 1.1 and 0.5 by 1.1 and adds the

products. This is his method for dealing with the multiplication of a fractional number by a whole number. He continues his working obtaining 179.685, 197.6535 and 217.41885. He reads some of the numbers stressing the values of the decimals.

4 To reduce 13 by one-tenth, five successive times, he starts by changing 13 to 130-tenths and deducts one-tenth thereof, i.e. 13, leaving 117. This he transforms to 1170 hundreds, to deduct 117 thereof. He goes on until he finally has left 7.67637, which he reads as 7 and 67637 parts of a hundred thousand.

GREEK INFLUENCE IN ARABIC ARITHMETIC

Other mathematicians from the ninth century followed and turned into Arabic all Greek scientific lore they could find, Hellenic, Hellenistic, Roman or even Byzantine. Much of this was geometrical; the main arithmetical works were parts of the *Elements* of Euclid, the *Introduction to Arithmetic* of Nicomachus of Gerasa (fl. *c.* AD 100), works of Hero of Alexandria (fl. AD 62) and *On the Measurement of the Circle* of Archimedes (287–212 BC).

Our objective here is to point out the development of this science on one particular example: the sequence of numbers.

Several types of progressions appear in Arabic texts, with rules for the *n*th term and the sum of the first *n* terms. The origin is definitely Greek. The Indians dealt with progressions. But immediately the Muslims understood the characteristics of Greek science, they preferred it to all other systems, especially the idea that it accepts only what can be proved logically, whereas the other systems give generally instructions and these are obeyed. The Muslims were so fond of proofs that they had distinct philosophies, or themes of thought, that we can translate as whyism, howism and whichism.

Some arithmetical and geometrical progressions, especially the duplicative $\Sigma 2^r$, appear in many books on arithmetic. But the following can be distinguished for comprehensive presentation:

1 *Al-Takmila* of Ibn Ṭāhir al-Baghdādī gives rules for the summation of $\Sigma r^2, r^3, r^4$ and polygonal numbers;

2 the *Marāsim* of al-Umawī[7] gives the same, but is more comprehensive despite being more compact;

3 the *Miftāḥ al-ḥisāb* (*Dictionary of Scientific Biography,* vol. VII, pp. 531–3) of al-Kāshī gives fifty rules described as useful for reckoners. These comprise most of the rules given by the preceding two works, although sometimes better worded. As usual, the author claims them as his own findings, though some of them go back to Euclid.

Below we summarize the contents of *al-Takmila* in summations, and add what else the *Marāsim* gives. Ibn Ṭāhir gives the following:

1 $$\sum_{1}^{n} m = \frac{1}{2}(n^2 + n) = \frac{n}{2}(n+1)$$ and

$$\sum_{a}^{m} m = (a+n)\cdot\frac{1}{2} \text{ number of terms}$$

$$= (a+n)\cdot\frac{1}{2}(n-a+1)$$

2 $1+3+5+\cdots+l = \left(\frac{1}{2}l+\frac{1}{2}\right)^2$, l being the last odd term.

3 $2+4+6+\cdots+l = \left(\frac{1}{2}l\right)^2+\frac{1}{2}l$, l being the last even number.

4 $2^2+2^3+\ldots$up to $2n$ terms$=(2^{n+1})^2-4$

$2^2+2^3+\ldots$up to $2n+1$ terms$=4(2^{n+1}-1)$, i.e.

$$\sum_{0}^{n} 4.2^r = 4(2^{n+1}-1)$$

5 $2.1+2.3+2.5+\ldots$up to n terms$=2n^2$, the nth term being $2(2n-1)$.

6(a) $$\sum_{1}^{n} m^2 = n(n+1)\left(\frac{1}{3}n+\frac{1}{6}\right) = n\left(n+\frac{1}{2}\right)(n+1)\cdot\frac{1}{3}$$

$$= (n^2+n)\left(\frac{n}{3}+\frac{1}{6}\right)$$

$$= \frac{n(n+1)(2n+1)}{6}$$

6(b) $$\frac{\sum_{1}^{n} m^2}{\sum_{1}^{n} m} = \frac{2}{3}n+\frac{1}{3}$$

7 $\sum r^3=(\sum r)^2$

8 Polygonal numbers: Consider the general arithmetic progression

$1, (1+a), (1+2a),\ldots, 1+(n-1)a$ (1)

The general term is $1+(n-1)a$, and the sum of n terms is $n+\frac{1}{2}n(n-1)a$. Giving a values like 1, 2, 3, 4 yields different arithmetical progressions,

like 1, 2, 3, 4,… natural numbers

1, 3, 5, 7,… odd numbers

1, 4, 7, 10,…

1, 5, 9, 13,…

Now we add up the terms of (1) in succession and obtain

1, (2+*a*), (3+3*a*), (4+6*a*) (2)

This is another series. We can easily see that its *n*th term is the sum of *n* terms of series $n+\frac{1}{2}n(n-1)a$. The Greeks called its terms polygonal numbers.

Giving *a* values like 1, 2, 3, 4, yields

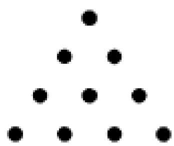
1, 3, 6, 10,… (2a)

1, 4, 9, 16,… (2b)

1, 5, 12, 22,… (2c)

1, 6, 15, 28,… (2d)

The idea is Greek and goes back to Pythagoras (sixth century BC). As in all mathematics, the Greek notion here is geometrical. Set (2a) was supposed to have arisen from structures like



Thus its elements are called triangular numbers. Likewise, set (2b) gives square numbers: set (2c) gives pentagonal numbers; set (2d) gives hexagonal numbers; and so on. So far so good, but what is the general term in each of these sets? We have to find the general term in sequence (2). The problem is to find

$$\sum_{1}^{n}\left[r+\frac{1}{2}r(r-1)a\right].$$

This is

$$\sum_{1}^{n}\left[r+\frac{1}{2}r(r-1)a\right]=\frac{1}{2}n(n+1)+\frac{1}{6}(n-1)^2n(n+1)a$$

$$=\frac{1}{2}n(n+1)\left[1+\frac{1}{3}(n-1)a\right]$$

a=1 gives the sum of *n* triangular numbers, i.e. $\frac{1}{2}n(n+1)(\frac{1}{3}n+\frac{2}{3})$;
a=2 gives the sum of the squares; and so on.

In *al-Takmila,* Ibn Ṭāhir gives, rhetorically of course, rules to find the sum of *n* terms of triangular, square, pentagonal and other polygonal numbers.

9 $$\sum_{1}^{n}m^4=\sum_{1}^{n}m^2\left[\frac{1}{5}\left(\sum_{1}^{n}m-1\right)+\sum_{1}^{n}m\right]$$

$$=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

10 Pyramidal numbers. The Greeks added the terms of sets (2a), (2b) etc. in succession to get new sets which they called pyramidal numbers. Thus, if we add the terms of set (2), we get

1, $(3+a)$, $(6+4a)$, $(10+10a)$… (3)

This is the series of pyramidal numbers.

$a=1$ gives 1, 4, 10, 20,…	the solid triangular sequence	(3a)
$a=2$ gives 1, 5, 14, 30,…	the solid square sequence	(3b)
$a=3$ gives 1, 6, 18, 40,…	the solid pentagonal sequence	(3c)
$a=4$ gives 1, 7, 22, 50,…	the solid hexagonal sequence	(3d)

and so on.

Ibn Ṭāhir dealt with such sequences and developed, amongst others, the following results:

(i) the nth term in (3c) is $\frac{3}{2}n^2-\frac{1}{2}n$,

(ii) the nth term in (3d) is $2n^2-n$.

11 Relations between polygonal numbers. Ibn Ṭāhir gives the following relations:

(i) The nth square=the nth triangle+the $(n-1)$th triangle, i.e.

$$n^2=\frac{1}{2}n(n+1)+\frac{1}{2}n(n-1)$$

(ii) The nth pentagon=the nth square+the $(n-1)$th triangle, i.e.

$$\frac{3}{2}n^2-\frac{1}{2}n=n^2+\frac{1}{2}n(n-1)$$

(iii) The nth hexagon=the nth square+twice the $(n-1)$th triangle, i.e.

$$2n^2-n=n^2+n(n-1)$$

(iv) Generally, the nth polygon−the $(n-1)$ polygon$=(n-1)a+1$.

The Greeks initially came up with the idea, but Ibn Ṭāhir developed it a lot further. Al-Umawī went farther still; he found out the sum of sequence (3). This is

$$S=\frac{1}{6}n(n+1)(n+2)+\frac{1}{24}(n-1)n(n+1)(n+2)a$$

$$=\frac{1}{6}n(n+1)(n+2)\,[1+\frac{1}{4}(n-1)a]$$

Thus giving a the proper values yields the sum of sequence (3a), (3b) etc.

Al-Umawī summarizes the rules of polygonal and pyramidal numbers as follows:

(i) in polygonal sequences, the nth term is $\frac{1}{2}n\,[2+(n-1)a]$ and the sum

$$S_n=\tfrac{1}{6}n(n+1)\,[3+(n-1)a];$$

(ii) in pyramidal sequences, the nth term is $\frac{1}{6}n(n+1)\,[3+(n-1)a]$ and

$$S_n=\tfrac{1}{24}\,n(n+1)(n+2)\,[4+(n-1)a].$$

He classifies all progressions as follows, giving the general term and the sum for each.

(i) Numerical, where the terms have a constant difference.

(ii) Natural, where the constant difference is 1.

(iii) Geometrical, where the terms have a constant ratio.

(iv) Duplicative, where the constant ratio is 2.

(v) Figurate, i.e. polygonal and pyramidal numbers.

(vi) Sequences like $r(r+1)$ which he calls stepping up sequences.

In connection with the stepping up sequence, he gives the following rules:

(i) $$1.2+2.3+\cdots+n(n+1)=\frac{1}{3}\,n(n+1)(n+2)$$

(ii) $$1.3+3.5+5.7+\cdots+N(N+2)=\frac{1}{3}\,N\left(\frac{N+2}{2}\right)(N+4)+\frac{1}{2}$$

N being odd

(iii) $$2.4+4.6+6.8+\cdots+M(M+2)=\frac{1}{3}\,M\left(\frac{M+2}{2}\right)(M+4)$$

M being even

We can express these rules as follows:

$$1.3+3.5+5.7+\cdots+(2n-1)(2n+1)=\frac{1}{6}\,(2n-1)(2n+1)(2n+3)+\frac{1}{2}$$

and

$$2.4+4.6+6.8+\cdots\ 2n(2n+2)=\frac{4}{3}\,n(n+1)(n+2)$$

Al-Kāshī treats almost the same types of progressions, but with a clearer idea, a better grasp of the subject and more effective generalization.

Some historical points are appropriate at this point.

The chapter on addition in the *Patiganita* is concerned exclusively with progressions. In a similar chapter on addition, al-Umawī deals with progressions. Hindu mathematics include summations of 2^r, r^2, r^3 and $\frac{1}{2}r(r+1)$ with various combinations of these.

On the other hand, Babylonian problems on progressions are well attested.

Again, the Greeks gave rules for adding geometrical progressions. As for polygonal numbers, Hypsicles in 175 BC defined them. Theon of Smyrna (second century AD) gave the rule for Σr^2 and some polygonal progressions. Nicomachus (*c.* 100 AD) deals with polygonal numbers in a way which highlights the influence of Ibn Ṭāhir's *al-Takmila*. Diophantus wrote a book on polygonal numbers, parts of which have survived.

Although Nicomachus makes a passing mention of pyramidal numbers, lamblichus (*c.* 283–330 AD) treated both polygonal and pyramidal numbers at length.

Ibn Ṭāhir and al-Umawī seem to be drawing from Greek fountains. How much of the material they present is original is difficult to ascertain. However, the Greek results as presented by Dickson (1919: vol. II, p. 4) make one think that the Muslims studied sequences in their own way. Anyhow, even if Islamic creation in this field is minor, the mere fact that they fetched things, put them together and

presented them to the world as a vivid integrated entity complete in itself and open for further development is an achievement worthwhile appreciating.

The above conclusion may also apply to other topics such as ratio and proportion and the calculus of irrationals. I shall devote the remaining part of the space allotted to me to giving some medieval arithmetic problems raised by the mathematicians to stimulate thinking and provide fun. These usually come in dramatic contexts; but they will be presented here as sheer arithmetic in modern terms.

Problem 1. This problem is by Ibn Ṭāhir from his *al-Takmila*. To find a hidden number $N \leq 105$, let a, b, c be such that

$$N \equiv a \pmod 5 \equiv b \pmod 7 \equiv c \pmod 3).$$

If these are made known to you, the number required is $21a+15b+70c-105k$, where k is any integer that makes the result ≤ 105.

Note that:

1 $21a+15b+70c-105k \equiv a \pmod 5 \equiv b \pmod 7 \equiv c \pmod 3$; i.e. it satisfies the conditions.

2 $21=3.7 \equiv a \pmod 5$; $15=3.5 \equiv b \pmod 7$; $70=\text{double } 7.5 \equiv c \pmod 3$. In explaining his method, the author says that to find a hidden number N take two numbers like 10 and 13 having no common factor. Take a, b where $N \equiv a \pmod{10} \equiv b \pmod{13}$. Then the number $N \leq 10.13=130$ can be found, it is

$$13ma+10nb-130k$$

m must satisfy $13m \equiv 1 \pmod{10}$; by inspection $m=7$, $13m=91 \equiv 1 \pmod{10}$. n must satisfy $10n \equiv 1 \pmod{13}$; by inspection $n=4$, $10n=40 \equiv 1 \pmod{13}$. Therefore, $N=91a+40b-130k$.

The above problem is obviously on arithmetic congruence. The concept appeared early in the Arabic world and was utilized in checking results, especially by casting out nines. According to Needham (1959:119), in a Chinese book going back to the fourth century AD there is the question of how to find a number that leaves 2 when divided by 3, 3 when divided by 5 and 2 when divided by 7. The solution has much in common with Ibn Ṭāhir's method. But against the claim that the idea was imported from China, one finds that in the first century AD, Nicomachus of

Gerasa worked on a similar problem; so did Brahmagupta in the seventh century.

It seems that puzzles circulate easily and attract many people in different places. Different people solve them in like or unlike methods. The following two problems have faced the author not less than ten times in the last sixty years, given in dramatic contexts.

Problem 2. Find the least number of weights that will weigh from 1 to 40 units, and the value of each weight.

The answer is four weights of 1, 3, 9 and 27 units. Obviously 1, 3 weigh up to 1+3 units; 1, 3, 9 weigh up to 1+3+9 units; and 1, 3, 9, 27 weigh up to 1+3+9+27 units, i.e. 40. The only manuscript that is so far attested to give the problem is one by Ibn Ghāzī al-Miknāsī.[8]

Problem 3. A judge has to divide seventeen camels amongst three partners A, B and C, so that A takes half the lot, B one-third, C one-ninth and the rest goes to the judge, with no camel being slaughtered and no two people sharing one camel.

The judge adds his camel to the lot making eighteen. A takes nine, B takes six, C takes two and the judge regains his camel. The solution is not mathematical. But why should it be? Is it not to the satisfaction of all?

NOTES

1 Cf. Vogel (1963). See also the chapter on the influence of Arabic mathematics.

2 *Al-Fihrist* by Ibn al-Nadīm (Arabic). There are several prints of this work; the one used by the present writer is an old undated print published in Cairo.

3 The full name of the book is *What Scribes, Employees and Others Need to Know of the Profession of Arithmetic*. Because it is in seven chapters, a shorter name is given to it, namely *The Book of Seven Grades* *(Kitāb al-Manāzil al-Sab')*.

4 Al-Karajī, known also as al-Karkhī, died about 1016. His *al-Kāfī fī al-ḥisāb (The Sufficient)* is being edited with a translation into German, at the Institute for the History of Arabic Science, Aleppo University, Syria.

5 See the chapter on algebra and al-Ṭūsī, *Œuvres Mathématiques*.

6 See chapter on further numerical analysis.

7 This is Ya'īsh ibn Ibrāhīm al-Umawī, an Andalusian who lived in Damascus (fourteenth century).

8 Ibn Ghāzī al-Miknāsī al-Fāsī (of Fez). His book is *Bughyat al-Ṭullāb fī Sharḥ Minyat al-Ḥussāb*; it is a commentary of an earlier work written in verse.

11
Algebra

ROSHDI RASHED

THE BEGINNING OF ALGEBRA: AL-KHWĀRIZMĪ

The 'publication' of the book of al-Khwārizmī at the beginning of the ninth century —between 813 and 833[1]—is an outstanding event in the history of mathematics. For the first time, one could see the term algebra appearing in a title[2] to designate a distinct mathematical discipline, equipped with a proper technical vocabulary. Muḥammad ibn Mūsā al-Khwārizmī, mathematician, astronomer and distinguished member of the 'House of Wisdom' of Baghdad, had compiled, he wrote, 'a book on algebra and *al-muqābala,* a concise book recording that which is subtle and important in calculation'.[3] The event was crucial, and was recognized as such by both ancient and modern historians. Its importance did not escape the mathematical community of the epoch,[4] nor that of the following centuries. This book of al-Khwārizmī did not cease being a source of inspiration and the subject of commentaries by mathematicians, not only in Arabic and Persian, but also in Latin and in the languages of Western Europe until the eighteenth century. But the event appeared paradoxical: to the novelty of the conception, of the vocabulary and of the organization of the book of al-Khwārizmī was contrasted the simplicity of the mathematical techniques described, if one compares them with the techniques in the celebrated mathematical compositions, of Euclid or Diophantus, for example. But this technical simplicity stems precisely from the new mathematical conception of al-Khwārizmī. Whilst one of the elements of his project was found twenty-five centuries before him with the Babylonians, another in the *Elements* of Euclid, a third in the *Arithmetica* of Diophantus, no earlier writer had recompiled these elements, and in this manner. But which are these elements, and what is this organization?

The goal of al-Khwārizmī is clear, never conceived of before: to elaborate a theory of equations solvable through radicals, which can be applied to whatever arithmetical and geometrical problems, and which can help in calculation, commercial transactions, inheritance, the surveying of land etc. In the first part of his book, al-Khwārizmī begins by defining the basic terms of this theory which, because of the requirement of resolution by radicals and because of his know-how in this area, was only concerned with equations of the first two degrees. In fact it is about the unknown, casually denoted by *root* or *thing,* its square, rational positive numbers, the laws of arithmetic ±, × / ÷, √, and equality. The principal concepts introduced next by al-Khwārizmī are the equation of the first degree, the equation of the second degree, the binomials and the associated trinomials, the normal form, algorithmic solutions, and the demonstration of the solution formula. The concept of equation appeared in the book of al-Khwārizmī to designate an infinite class of problems, and not, as with the Babylonians for example, in the course of the solution of one or other problem. However, the equations are not presented in the course of the solution of problems to solve, like the ones of the Babylonians and Diophantus, but from the start, from the basic terms whose combinations must give all the possible forms. Thus, al-Khwārizmī, immediately after having introduced the basic terms, gives the six following types:

$$ax^2=bx \qquad ax^2=c \qquad bx=c$$

$$ax^2+bx=c \qquad ax^2+c=bx \qquad ax^2=bx+c$$

He then introduces the notion of normal form, and needs to reduce each of the preceding equations to the corresponding normal form. He finds in particular, for the trinomial equations,

$$x^2+px=q \qquad x^2=px+q \qquad x^2+q=px \tag{1}$$

Al-Khwārizmī next passed to the determination of algorithmic formulae for the solutions. He treated each case, and obtained formulae equivalent to the following expressions:

$$x=\left[\left(\frac{p}{2}\right)^2+q\right]^{1/2}-\frac{p}{2}$$

$$x=\frac{p}{2}+\left[\left(\frac{p}{2}\right)^2+q\right]^{1/2}$$

$$x=\frac{p}{2}\pm\left[\left(\frac{p}{2}\right)^2-q\right]^{1/2} \qquad \text{if } \left(\frac{p}{2}\right)^2>q$$

and in this last case he clarifies[5]

If $\left(\frac{p}{2}\right)^2=q$ ‘then the root of the square *[māl]* is equal to half of the the number of roots, exactly, without surplus or diminution’

If $\left(\frac{p}{2}\right)^2<q$ ‘then the problem is impossible’

Al-Khwārizmī also demonstrates different formulae, not algebraically, but by means of the idea of equality of areas. He was probably inspired by a very recent knowledge of the *Elements* of Euclid, translated by his colleague at the House of Wisdom, **al-Ḥajjāj** ibn **Maṭar**. Each of these demonstrations is presented by al-Khwārizmī as the ‘cause’—***ʿilla***—of the solution. Also al-Khwārizmī not only required each case to be demonstrated, but he proposed sometimes two demonstrations for one and the same type of equation. One such requirement marks well the distance covered, and not only separates al-Khwārizmī from the Babylonians, but also, by his systematic working from now on, from Diophantus.

Thus, for example, for the equation $x^2+px=q$, he takes two segments AB=AC=x and then takes CD=BE=$p/2$ (Figure 11.1). If the sum of the surfaces ABMC, BENM, DCMP is equal to q, the surface of the square AEOD is equal to $(p/2)^2+q$, whence[6]

$$x=\left[\left(\frac{p}{2}\right)^2+q\right]^{1/2}-\frac{p}{2}$$

With al-Khwārizmī, the concepts of the new discipline, and notably 'the thing', the unknown, are not designated to be a particular entity but an object which can be either numerical or geometrical; on the other hand the algorithms of the solution must be themselves an object of demonstration. It is there that the principal elements of the contribution of al-Khwārizmī reside. As he saw it, all problems dealt with from now on in algebra,

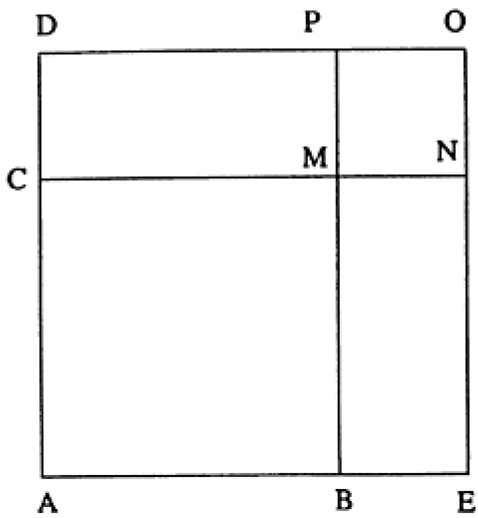


Figure 11.1

whether they be arithmetic or geometry, must be reduced to an equation with a single unknown and with positive rational coefficients of second degree at most. The algebraic operations—transposition and reduction—are then applied to put the equation in normal form, which makes possible the idea of a solution as a simple procedure of decision, an algorithm for each class of problems. The formula of the solution is then justified mathematically, with the help of a proto-geometric demonstration, and al-Khwārizmī is in a position to write that everything found in algebra 'must lead you to one of the six types that I described in my book'.[7]

Al-Khwārizmī then undertakes a brief study of some properties of the application of elementary laws of arithmetic to the simplest algebraic expressions. He studies in this way products of the type

$$(a \pm bx)(c \pm dx) \quad \text{with } a, b, c, d \in Q_+$$

As rudimentary as it appears to be, this study represents no less than the first attempt at algebraic calculation as such, since the elements of this calculation became the subject of relatively autonomous chapters. These are then followed by other chapters in which al-Khwārizmī proceeds to the application of a worked out theory, in order to solve numerical and geometrical problems, before treating at last the problems of inheritance with the aid of algebra, in which he comes across some problems of indeterminate analysis.

Thus, at first, algebra is presented as a kind of arithmetic, more general than the 'logistic'—because it allows 'logistic' problems to be solved more rigorously thanks to these concepts—but also more general than metric geometry. The new discipline is in fact a theory of linear and quadratic equations with a single unknown solvable by radicals, and of algebraic calculation on the associated expressions, without yet the concept of a polynomial.

THE SUCCESSORS OF AL-KHWĀRIZMĪ AND THE DEVELOPMENT OF ALGEBRAIC CALCULATION

In order to grasp better the idea that al-Khwārizmī developed in the new discipline, as well as its fruitfulness, it is certainly insufficient to compare his book with ancient mathematical compositions; it is also necessary to examine the impact that he had on his contemporaries and on his successors. It is only then that he rises up in his true historical dimension. One of the features of this book, essential to our minds, is that it immediately aroused a trend of algebraic research. The biobibliographer of the tenth century, al-Nadīm, has delivered us already a long list of contemporaries and of successors of al-Khwārizmī who followed his research. Amongst many others were Ibn Turk, Sind ibn ʿAlī, al-Ṣaydanānī, Thābit ibn Qurra, Abū Kāmil, Sinān ibn al-Fatḥ, al-Ḥubūbī and Abū al-Wafāʾ al-Būzjānī. Although a good number of their writings have disappeared, enough have reached us to restore the main lines of this tradition, but it is not possible for us within the limits of this chapter to take up an analysis of each of the contributions. We attempt only to extract the principal axes of the development of algebra following al-Khwārizmī.

In the time of al-Khwārizmī and immediately afterwards, we witness essentially the expansion of research already begun by him: the theory of quadratic equations, algebraic calculation, indeterminate analysis and the application of algebra to problems of inheritance, partition etc. Research into the theory of equations was down several avenues. The first was that already opened up by al-Khwārizmī himself, but this time with an improvement of his proto-geometric demonstrations: it is the path followed by Ibn Turk[8] who, without adding anything new, reproduced a tighter discussion of the proof. More important is the path that Thābit ibn Qurra took a little later. He comes back to the *Elements* of Euclid, both to establish the demonstrations of al-Khwārizmī on more solid geometrical bases and to explain equations of second degree geometrically. Moreover, Ibn Qurra is the first to distinguish clearly between the two methods, algebraic and geometrical, and he seeks to show that they both lead to the same result, i.e. to a geometrical interpretation of algebraic procedures. Ibn Qurra begins by showing that the equation $x^2+px=q$ can be solved with the help of proposition II.6 of the *Elements.* At the end of his proof, he writes: 'this method corresponds to the method of the algebraists—**aṣḥāb** *al-jabr'.*[9] He continues with $x^2+q=px$ and $x^2=px+q$, with the help respectively of II.5 and II.6 of the *Elements;* he shows for each the correspondence with the algebraic solutions, and writes: 'The method for solving this problem and the one that precedes it by geometry is the way of its solution by algebra'.[10] The mathematicians subsequently confirmed these conclusions. One of them writes: 'It was shown that the procedure which led to the determination of the sides of the unknown squares in each of three trinomial equations is the procedure given by Euclid at the end of the sixth book of his work on the *Elements,* and which is to apply to a given straight line a parallelogram which exceeds the whole parallelogram or which is deficient by a square. The side of the square in excess is the side of the unknown square in the first and second trinomial equations $(x^2+q=px$, $x^2+px=q)$, and in the third trinomial equation it is the sum of the straight line on which the parallelogram is applied and the side of the square in excess'.[11]

But this geometrical explanation by Ibn Qurra of the equations of al-Khwārizmī proved to be particularly important, as we shall see, in the development of the theory of algebraic equations. Another account, very different, appeared at nearly the same time, and it also would be fundamental for the development of the theory of algebraic equations: the explanation of geometrical problems in algebraic terms. Indeed al-Māhānī, a contemporary of Ibn Qurra, began not just the translation of certain biquadratic problems in book X of the *Elements* into algebraic equations, but also a problem on solids, given in Archimedes' *The Sphere and the Cylinder,* in a cubic equation.[12]

Another direction of development of the theory of equations followed at the time was research on the general form of the equations

$$ax^{2n}+x^n=c \qquad ax^{2n}+c=bx^n \qquad ax^{2n}=bx^n+c$$

as we can establish in the work of Abū Kāmil and Sinān ibn **al-Fatḥ,** amongst others.

Furthermore we witness, after al-Khwārizmī, the expansion of algebraic calculation. This, perhaps, is the principal theme of research, and the one more communally shared, amongst the algebraists following him. Thus even the terms of algebra have begun to be extended up to the sixth power of the unknown, as can be seen in the work of Abū Kāmil and Sinān ibn **al-Fatḥ.** Furthermore the latter[13] defines powers multiplicatively, in contrast with Abū Kāmil who gives an additive definition. But it is the algebraic work of Abū Kāmil which marks both this period and the history of algebra.[14] In addition to the expansion of algebraic calculation, he included into his book a new area of algebra, indeterminate analysis or rational Diophantine analysis. Thus, after having taken up the theory of equations with firmer demonstrations than those of his predecessor, he studies in a much more thorough and extensive manner the arithmetical operations on binomials and trinomials, demonstrating the result obtained each time. He states and justifies the sign rule and establishes calculation rules for fractions before passing to systems of linear equations with several unknowns and equations with irrational coefficients, such as

$$\left(x^2+\frac{1}{\sqrt{2}}\,x\right)^2=4x^2 \qquad \frac{\sqrt{10x}}{(2+\sqrt{3})}=x-10$$

Abū Kāmil integrates into his algebra the auxiliary numerical methods, of which some would have been contained in a lost book of al-Khwārizmī, such as

$$\sum_{k=1}^{n} a_k \qquad \sum_{k=1}^{n} k^2 \qquad \sum_{k=1}^{n} 2k$$

Abū Kāmil then studied numerous problems which lead to second degree equations.

We thus see that the research of al-Khwārizmī's successors, and especially Abū Kāmil, contributed to the theory of equations and to the extension of algebraic calculation to the field of rational numbers, and to the set of irrational numbers. The research of Abū Kāmil on indeterminate analysis had considerable repercussions in the development of this field, but it also gave him a new signification and a new status. Part of algebra, this analysis constitutes from now on a subject area in all treatments intended to cover the discipline.

THE ARITHMETIZATION OF ALGEBRA: AL-KARAJĪ AND HIS SUCCESSORS

We shall understand nothing of the history of algebra if we do not emphasize the contributions of two research movements which developed during the period previously considered. The first was directed towards the study of irrational quantities, whether as a result of a reading of book X of the *Elements* or in a way independently. We can recall, amongst many other mathematicians who took part in this research, the names of al-Māhānī, Sulaymān ibn ʿIṣma, al-Khāzin, al-Aḥwāzī, Yūḥannā ibn Yūsuf, al-Hāshimī and so on. It goes without saying that we cannot cover here these various contributions. We would just like to emphasize that, in the course of this work, calculation with irrational quantities was actively developed, and sometimes even parts of book X of the *Elements* of Euclid started being read in the light of the algebra of al-Khwārizmī. To take a single example, we consider that of al-Māhānī in the ninth century, who searched for the square root of five apotomes. Thus, to extract the square root of the first apotome,[15] al-Māhānī suggested that 'we proceed by the method of algebra and *al-muqābala*',[16] i.e. putting $a=x+y$ and $b=4xy$, one obtains the equation $x^2+b/4=ax$. One then determines the positive root x_0, deduces y_0 and obtains

$$(a-\sqrt{b})^{½}=\sqrt{x}0-\sqrt{y}0$$

Al-Māhānī carries on thus for the next four apotomes, and for the second apotome *(√b–a)* for example, with b=45 and a=5, he ends up with the equation

$$x^4 + \frac{625}{16} = \frac{65}{2} x^2$$

Now these mathematicians not only studied algebraic calculation of irrational quantities, but they were also to confirm the generality of algebra as a tool.

The second research movement was instigated by the translation of the *Arithmetica* of Diophantus into Arabic, and notably by the algebraic reading of this last book. It is about 870 when Qusṭā ibn Lūqā translates seven books of Diophantus's *Arithmetica* under the significant title *The Art of Algebra*.[17] The translator used the language of al-Khwārizmī to reproduce the Greek of Diophantus, thus reorienting the contents of the book towards the new discipline. Now the *Arithmetica,* even if they were not a work on algebra in the sense of al-Khwārizmī, nevertheless contained techniques of algebraic calculation that were powerful for the time: substitution, elimination, changing of variables etc. They were the subject of commentaries by mathematicians such as Ibn Lūqā, their translator, in the ninth century, and Abū al-Wafā' al-Būzjānī a century later, but these texts are unfortunately lost. We know, however, that al-Būzjānī wanted to prove the Diophantine solutions in his commentary. This same Abū al-Wafā', in a text which is available to us, demonstrates the binomial formula, often used in the *Arithmetica,* for n=2.3.[18]

Be that as it may, the progress of algebraic calculation, whether by its expansion into other areas or by the mass of technical results obtained, succeeded in rejuvenating the discipline itself. A century and a half after al-Khwārizmī, the Baghdad mathematician al-Karajī thought of another research project: the application of arithmetic to algebra, i.e. to study systematically the application of the laws of arithmetic and of certain of its algorithms to algebraic expressions and in particular to polynomials. It is precisely this calculation on the algebraic expressions of the form

$$f(x) = \sum_{k=-m}^{n} a_k x^k \qquad m, n \in Z_+$$

which has become the principal aim of algebra. The theory of algebraic equations is of course always present, but only occupies a modest place in the preoccupations of algebraists. One understands from that time on, that books on algebra undergo modifications not only in their content but also in their organization.

Al-Karajī devoted several writings to this new project, notably *al-Fakhrī* and *al-Badī'*. These books will be studied, reproduced and commented on by mathematicians until the seventeenth century, i.e. that the work of al-Karajī occupied the central place in research on arithmetical algebra for centuries, whilst the book of al-Khwārizmī became a historically important exposé, commented on only by second rate mathematicians. Without reproducing here the history of six centuries of algebra, we illustrate the impact of the work of al-Karajī by turning towards one of his successors in the twelfth century, al-Samaw'al (d. 1174). The latter includes in his algebra, *al-Bāhir,* the principal writings of al-Karajī and notably the two works cited previously. Al-Samaw'al begins by defining in a general way the notion of algebraic power[19] and, from the definition $x^0=1$, gives the rule equivalent to $x^m x^n = x^{m+n}$, $m, n \in Z$. Next comes the study of arithmetical operations on monomials and polynomials, notably those on the divisibility of polynomials, as well as the approximation of fractions by the elements of the ring of polynomials. We have for example

$$\frac{f(x)}{g(x)} = \frac{20x^2+30x}{6x^2+12} \approx \frac{10}{3} + \frac{5}{x} - \frac{20}{3x^2} - \frac{10}{x^3} + \frac{40}{3x^4} + \frac{20}{x^5} - \frac{80}{3x^6} - \frac{40}{x^7}$$

Al-Samaw'al obtains a sort of limited expansion of $f(x)/g(x)$ which is only valid for sufficiently large x.

We meet next the extraction of a square root of a polynomial with rational coefficients. But, in all the calculations on polynomials, al-Karajī had devoted a writing, lost now but luckily cited by al-Samaw'al, where he occupied himself with establishing the formula of binomial expansion and the table of its coefficients:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad n \in N$$

It is during the demonstration of this formula that complete finite induction appears in an archaic form as the procedure of mathematical proof. Amongst the methods of auxiliary calculation, al-Samaw'al, following al-Karajī, gives the sum of different arithmetic progressions, with their demonstration:

$$\sum_{k=1}^{n} k,\ \sum_{k=1}^{n} k^2,\ \left(\sum_{k=1}^{n} k\right)^2,\ \sum_{k=1}^{n} k(k+1),\ \ldots$$

Next comes the response to the following question: 'How can multiplication, division, addition, subtraction and the extraction of roots be used for irrational quantities?'[20] The answer to this question led al-Karajī and his successors to reading algebraically, and in a deliberate manner, book X of the *Elements,* to extend to infinity the monomials and binomials given in this book and to propose rules of calculation, amongst which we find explicitly formulated those of al-Māhānī,

$$(x^{1/n})^{1/m} = (x^{1/m})^{1/n} \quad \text{and} \quad x^{1/m} = (x^n)^{1/mn}$$

with others such as

$$(x^{1/m} \pm y^{1/m}) = \{y[(x/y)^{1/m} \pm 1]^m\}^{1/m}$$

We also find an important chapter on rational Diophantine analysis, and another on the solution of systems of linear equations with several unknowns. Al-Samaw'al gives a system of 210 linear equations with ten unknowns.

From the works of al-Karajī, one sees the creation of a field of research in algebra, a tradition recognizable by content and the organization of each of the works. These, to reproduce the words of Ibn al-Bannā' in the thirteenth and fourteenth centuries, 'are almost innumerable'.[21] Citing here only a few, we find the masters of al-Samaw'al: al-Shahrazūrī, Ibn Abī Turāb, Ibn al-Khashshāb; al-Samaw'al himself, Ibn al-Khawwām, al-Tanūkhī, Kamāl al-Dīn al-Fārisī, Ibn al-Bannā' and, later, al-Kāshī, al-Yazdī etc.

In the midst of this tradition, the theory of algebraic equations, strictly speaking, is not central but nevertheless makes some progress. Al-Karajī himself considered quadratic equations, just like his predecessors. Certain of his successors, however, attempted to study the solution of cubic and quartic equations. Thus al-Sulamī, in the twelfth century, tackled cubic equations to find a solution by radicals.[22] The text of al-Sulamī testifies to the interest of the mathematicians of his time in a solution of cubic equations by radicals. He himself considers two types as possible:

$$x^3+ax^2+bx=c \quad \text{and} \quad x^3+bx=ax^2+c$$

However, he imposes the condition $a^2=3b$, and then gives for each equation a positive real root:

$$x=\left(\frac{a^3}{27}+c\right)^{1/3}-\frac{a}{3} \qquad \text{and} \qquad x=\left(c-\frac{a^3}{27}\right)^{1/3}+\frac{a}{3}$$

We can reconstruct the procedure of al-Sulamī as follows: by affine transformation he obtains the equation in its normal form; but, instead of finding the discriminant, he leaves aside the coefficient of the first power of the unknown to change the problem to one of extraction of a cubic root. Thus, for example, for the first equation, we take the affine transformation $x \to y-a/3$; the equation can be rewritten as

$$y^3+py-q=0$$

with

$$p=b-\frac{a^2}{3} \qquad \text{and} \qquad q=c+\frac{a^3}{27}+\left(b\,\frac{a}{3}-\frac{a^3}{9}\right)$$

Putting $b=a^2/3$, we have

$$y^3 = c + \frac{a^3}{27}$$

whence *y,* and therefore *x.*

Such attempts, attributed to the fourteenth-century Italian mathematician Master Dardi,[23] are frequent in the algebraic tradition of al-Karajī.

Thus, for example, the mathematician Ibn al-Bannā',[24] even though he recognizes implicitly the difficulty in solving cubic equations by radicals with the exception of $x^3=a$ when he writes that, for equations which 'lead to other degrees (than the second), one cannot solve them by the method of algebra with the exception of "cubes equal to a number"', gives the equation

$$x^4+2x^3=x+30 \qquad (*)$$

which he solves in the following manner: one rewrites the equation

$$x^4+2x^3+x^2=x^2+x+30$$

which can be rewritten as

$$(x^2+x)^2=x^2+x+30$$

Putting $y=x^2+x$, one has

$$y^2=y+30$$

On solving this equation, one has $y=6$, and one can then solve $x^2+x=6$ to find $x=2$, a solution of (*).

It is still too early to know exactly the contribution of mathematicians of this tradition to the solution of cubic and quartic equations; but this evidence, contrary to what we might think, shows that certain of them attempted to go much further than al-Karajī.

THE GEOMETRIZATION OF ALGEBRA: AL-KHAYYĀM

The algebraist arithmeticians kept to the solution of equations by radicals, and wanted to justify the algorithm of the solution. Sometimes even, from the same mathematician, Abū Kāmil for example, we come across two justifications, one geometric and the other algebraic. For cubic equations, he was missing not only their solution by radicals, but equally the justification of the solution algorithm, because the solution cannot be constructed with a ruler and a compass. The mathematicians of this tradition were perfectly aware of this fact, and one had written well before 1185: ‘Since the unknown that one wants to determine and know in each of these polynomials is the side of the cube mentioned in each, and the analysis leads to the application of a known right-angled parallelepiped to a known line, and which is surplus to the entire parallelepiped by a cube or which is deficient by a cube; we can only do this synthese using conic sections’.[25] Now this recourse to conic sections, explicitly intended to solve cubic equations, quickly followed the first algebraic renderings of solid problems. We have mentioned in the ninth-century al-Māhānī and the lemma of Archimedes;[26] it was not long before other problems such as the trisection of an angle, the two means and the regular heptagon in particular were translated into algebraic terms. However, confronted with the difficulty mentioned above, and thus with that of solving cubic equations by radicals, mathematicians such as al-Khāzin, Ibn ‘Irāq, Abū al-Jūd ibn al-Layth, al-Shannī etc. ended up translating this equation into geometrical terms.[27] They then found in the course of studying this equation a technique already being used in the examination of solid problems, i.e. the intersection of conical curves. This is precisely the reason for the geometrization of the theory of algebraic equations. This time, in contrast with Thābit ibn Qurra, one does not look for a geometrical translation of algebraic equations to find the geometrical equivalent of the algebraic solution already obtained, but to determine, with the help of geometry, the positive roots of the equation that have not yet been found by other means. The attempts of al-Khāzin, al-Qūhī, Ibn al-Layth, al-Shannī, al-Bīrūnī etc. are just partial contributions until the conception of the project by al-Khayyām: the elaboration of a geometrical theory for equations of degree equal to or less than 3. Al-Khayyām (1048–1131) intended first to supersede the fragmentary research, i.e. the research linked in one or another form with cubic equations, in order to elaborate a theory of equations and to propose at the same time a new style of mathematical writing. Thus he studied all types of third degree equations, classed in a formal way according to the distribution of constant terms, of first degree, of second degree and of third degree, between the two members of the equation. For each of these types, al-Khayyām found a construction of a positive root by the intersection of two conics. Thus for example to solve the equation ‘a cube is equal to the sides plus a number’, i.e.

$$x^3=bx+c \qquad b, c>0 \tag{*}$$

al-Khayyām considered only the positive root. To determine it, he proceeded from the intersection of a semi-parabola

$$P=\{(x, y);\ b^{1/2}y=x^2\}$$

and a branch of an equilateral hyperbola having the same vertex:

$$H=\left\{(x,y); y^2=\left(\frac{c}{b}+x\right)x\right\}$$

He showed that they have a second common point which corresponds to the positive root. We note that, if one takes the parabola and the hyperbola, for certain values of *b* and *c,* the points of intersection which correspond to the negative roots, can be obtained (Figure 11.2).

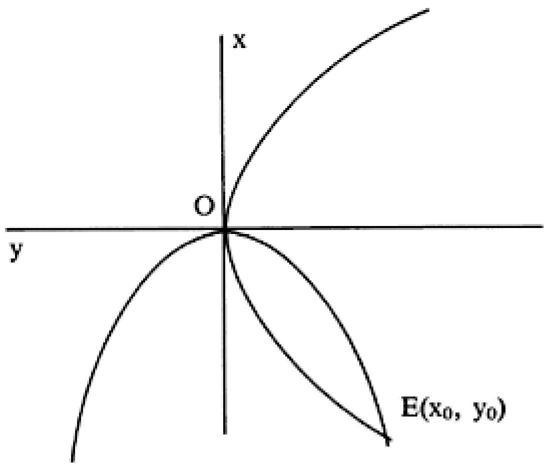


Figure 11.2

Thus, for the choice of curves, we note that if we introduce a trivial solution $x=0$ then equation (*) becomes

$$\frac{x^4}{b}=x^2+\frac{c}{b}x$$

whence we obtain the two preceding curves. From their intersection (x_0, y_0)

$$\frac{b^{1/2}}{x_0}=\frac{x_0}{y_0}=\frac{y_0}{x_0+c/b}$$

whence

$$\frac{b}{x_0^2}=\frac{x_0}{x_0+c/b}$$

and x_0 is the solution of equation (*).

To work out this new theory, al-Khayyām is forced to conceive and formulate new and better relations between geometry and algebra. We recall in this regard that the fundamental concept introduced by al-Khayyām is the unit of measure which, suitably defined with respect to that of dimension, allowed the application of geometry to algebra.

Now this application led al-Khayyām in two directions, which could seem at first to be paradoxical: whilst algebra was now identified with the theory of algebraic equations, this seemed from now on, but still hesitantly, to transcend the split between algebra and geometry. The theory of equations is more than ever a place where algebra and geometry meet and, more and more, analytical arguments and methods. The real evidence of this situation is the appearance of memoirs dedicated to the theory of equations, such as that of al-Khayyām. Contrary to algebraist arithmeticians, al-Khayyām moves away from his treatise the chapters on polynomials, on polynomial arithmetic, on the study of algebraic irrationals etc. He also creates a new style of mathematical writing: he begins with a discussion of the concept of algebraic magnitude, to define the concept of measure unit; he advances next the necessary lemmas as well as a formal classification of equations —according to the number of terms—before then examining, in order of increasing difficulty, binomial equations of second degree, binomial equations of third degree, trinomial equations of second degree, trinomial equations of third degree, and finally equations containing the inverse of the unknown. In his treatise, al-Khayyām reaches two remarkable results that historians have usually attributed to Descartes: a general solution of all equations of third degree by the intersection of two conics; and a geometrical calculation made possible by the choice of a unit length, keeping faithful, in contrast to Descartes, to the homogeneity rule.

Al-Khayyām, we note, does not stop there but tries to give an approximate numerical solution for cubic equations. Thus, in a memoir entitled *On the division of a quarter of a circle,*[28] in which he announces his new project on the theory of equations, he reaches an approximate numerical solution by means of trigonometrical tables.

THE TRANSFORMATION OF THE THEORY OF ALGEBRAIC EQUATIONS: SHARAF AL-DĪN AL-ṬŪSĪ

Until recently, it was thought that the contribution of mathematicians of this time to the theory of algebraic equations was limited to al-Khayyām and his work. Nothing of the kind. Not only did the work of al-Khayyām begin a real tradition but, in addition, it was profoundly transformed barely a half century after his death.

According to historical evidence the student of al-Khayyām, Sharaf al-Dīn al-Masʿūdī,[29] would have written a book which treated the theory of equations and the solution of cubic equations. But this book, if it was written, has not reached us at all. Two generations after al-Khayyām, we encounter one of the most important works of this movement: the treatise of Sharaf al-Dīn al-Ṭūsī *On the Equations*.[30] Now this treatise of al-Ṭūsī (about 1170) makes some very important innovations with respect to that of al-Khayyām. Unlike the approach of his predecessor, that of al-Ṭūsī is no longer global and algebraic but local and analytic. This radical change, particularly important in the history of classical mathematics, necessitates that we consider it a little longer.

The *Treatise* of al-Ṭūsī opens with the study of two conical curves, used in the following. It concerns a parabola and a hyperbola, to which is added a circle assumed known, to exhaust all the curves to which the author had recourse. He seems to suppose that his reader is familiar with the equation of a circle, obtained from the power of a point with respect to it, and uses this preliminary part to establish the equation of a parabola and the equation of an equilateral hyperbola, with respect to two systems of axes.

Next follows a classification of equations of degree less than or equal to 3. In contrast with al-Khayyām, he opts for an extrinsic criterion of classification rather than intrinsic. Whilst al-Khayyām, as we have noted, organizes his exposition according to the number of monomials which form the equation, al-Ṭūsī chooses as the criterion the existence or not of positive solutions; i.e. the equations are arranged according to whether they allow 'impossible cases' or not. One easily understands then that the *Treatise* is made up of only two parts, corresponding to the preceding alternatives. In the first part, al-Ṭūsī deals with the solution of twenty equations; for each case, he proceeds through a geometrical construction of roots, the determination of the discriminant for the only quadratic equations, and finally to the numerical solution with the help of the method known as Ruffini-Horner. He reserves the application of this method to polynomial equations, and not just to the extraction of the root of a number.

Already, we can therefore spot the constituent elements of the theory of equations of the twelfth century, in the tradition of al-Khayyām: geometrical construction of roots, numerical solution of equations, and finally recall of the solutions by radicals of the quadratic equation, this time rediscovered from geometrical construction. In the first part, after having studied second degree equations and the equation $x^3=c$, al-Ṭūsī examines eight third degree equations. The first seven all have a single positive root. They can have negative roots that al-Ṭūsī did not recognize. To study each of these equations, he chooses two second degree curves or, more precisely, two curved segments. He shows, through geometric means, that the arcs under study have a point of intersection whose abscissa verifies the proposed equation (they can have other points of intersection). The geometrical properties described by al-Ṭūsī are, aside from some details which he passes over though they are satisfied by the data he chooses, the characteristic properties and thus lead to the equations of the curves under consideration. Thanks to the use of the terms 'interior' and 'exterior', al-Ṭūsī can employ the continuity of the curves and their convexity. We can thus translate his approach to the equation

$$x^3-bx=c \qquad b, c>0$$

He considered in fact the two expressions

$$g(x)=\left[x\left(\frac{c}{b}+x\right)\right]^{1/2} \qquad \text{and} \qquad f(x)=\frac{x^2}{\sqrt{b}}$$

and showed that, if α and β exist such that $(f-g)(\alpha)>0$ and $(f-g)(\beta)<0$, then there exists $\gamma \in\,]\alpha, \beta[$ such that $(f-g)(\gamma)=0$.

In the reading of this first part, we see that, as with al-Khayyām, al-Ṭūsī studies principally the geometric construction of positive roots of the twenty equations of degree less than or equal to 3, since those which are left are transformed by means of affine transformations to one or other of these types. In an analogous method to that of al-Khayyām, he begins with plane geometrical constructions if the equation, reduced as much as possible, is of first or second degree and by constructions using two or three of the curves mentioned if the equation, reduced as much as possible, is cubic.

Although the first part of the *Treatise* is closely dependent on the contribution of al-Khayyām, one already perceives some differences, of which the consequences only appear in the second part. For each equation studied, al-Ṭūsī demonstrates the existence of a point of intersection of two curves, while al-Khayyām only really undertook this study for the twentieth equation. Al-Ṭūsī has also introduced some ideas which he will have recourse to frequently in the second part, such as affine transformations and the distance of a point from a line.

The second part of the *Treatise* is dedicated to five equations which, according to the expression of al-Ṭūsī, allow 'impossible cases', i.e. cases where there is no positive solution. They are the equations

(1) $x^3+c=ax^2$
(2) $x^3+c=bx$
(3) $x^3+ax^2+c=bx$
(4) $x^3+bx+c=ax^2$
(5) $x^3+c=ax^2+bx$

In contrast with al-Khayyām, al-Ṭūsī could not be content with a simple statement of these 'impossible cases'. Preoccupied with the proof of the existence of points of intersection, and consequently with the existence of roots, he had to characterize such cases and look for their justification. Now it is precisely the meeting of this technical problem and the questioning which followed which brought al-Ṭūsī to break from the tradition of al-Khayyām and to modify his initial project. But, to comprehend this major change, it is necessary to analyse the approach of al-Ṭūsī.

Each of the five equations are written in the form $f(x)=c$; f is a polynomial. To characterize the 'impossible cases', al-Ṭūsī studies in fact the intersection of the curve $y=f(x)$ with the line $y=c$. For al-Ṭūsī, it is a segment of the curve, that for which we have simultaneously $x>0$ and $y=f(x)>0$, a segment that may not exist. We note that, for him, the problem only makes sense if $x>0$ and $f(x)>0$, and in each case he poses the condition so that $f(x)$ is strictly positive. Thus, in equation (1) he poses the condition $0<x<a$, in equation (2) $0<x<\sqrt{b}$; in (3) he gives the condition $0<x<\sqrt{b}$, which is, however, not sufficient. Al-Ṭūsī is therefore constrained to examine the relationship between the existence of solutions and the position of the constant c with respect to the maximum of the polynomial function. It is on this occasion that he introduces new concepts, new procedures and a new language; and in addition, he defines a new subject. He thus begins by formulating the concept of the maximum of an algebraic expression, which he calls 'the largest number'—*al-ʿadad al-aʿẓam*. Let $f(x_0)=c_0$ be the maximum; this gives the point (x_0, c_0). Al-Ṭūsī determines next the roots of $f(x)=0$, i.e. the intersection of the curve with the abscissa; finally he deduces a double inequality for the roots of $f(x)=c$.

The whole problem from now on is therefore for him to find the value of x which yields a maximum of $f(x)$. Al-Ṭūsī proceeds then by solving an equation which turns out to be, though in a different notation, $f'(x)=0$, where f' is the polynomial, derivative of f. But, before examining this central problem of the derivative, we note the change and the introduction of local analysis. We begin by recalling the results of al-Ṭūsī. For equation (1) the derivative admits two roots, 0 and $2a/3$, which give respectively a minimum $f(0)=0$ and a maximum $f(2a/3)=c_0$. On the other hand, the equation $f(x)=0$ admits a double root $\lambda_1=0$ and a positive root $\lambda_2=a$. Al-Ṭūsī therefore concludes: if $c<c_0$ equation (1) has two positive roots x_1 and x_2 such that $\lambda_1=0<x_1<x_0<x_2<\lambda_2=a$. Notice that a third, negative, root x_3 exists, which al-Ṭūsī did not consider. For equations (2), (3) and (5) his reasoning is analogous. In these three cases, the derivative admits two roots with opposite signs. The positive root x_0 gives the maxwhich one is negative and the others are $\lambda_1=0$ and λ_2—whence the concluimum $c_0=f(x_0)$ and the equation $f(x)=0$ admits three simple roots, of sion obtained previously. To best illustrate the approach of al-Ṭūsī, we resume his discussion of equation (1). This equation can be rewritten

$$c=x^2(a-x)=f(x)$$

Al-Ṭūsī considered three cases.

$$c>\frac{4a^3}{27}$$

The problem is impossible, according to al-Ṭūsī (it admits a negative root).

$$c=\frac{4a^3}{27}$$

Al-Ṭūsī determines the double root $x_0=2a/3$ (but does not recognize the negative root).

$$c<\frac{4a^3}{27}$$

Al-Ṭūsī determines two positive roots, with

$$0<x_1<\frac{2a}{3}<x_2<a$$

He then studies the maximum of $f(x)$; he shows that

$$f(x_0) = \sup_{0<x<a} f(x) \qquad \text{with } x_0 = \frac{2a}{3} \qquad (*)$$

by first showing that

(a) $$x_1 > x_0 \Rightarrow f(x_1) < f(x_0)$$

followed by

(b) $$x_2 < x_0 \Rightarrow f(x_2) < f(x_0)$$

and from (a) and (b) he gets (*).

To find $x_0=2a/3$, al-Ṭūsī solves $f'(x)=0$. He next calculates

$$f(x_0) = f\left(\frac{2a}{3}\right) = \frac{4a^3}{27}$$

which allows him to justify the three cases considered previously. He next determines the two positive roots x_1 and x_2. He puts $x_2=x_0+y$; this affine transformation leads to the equation

$$y^3+ay^2=k$$

with $k=c_0-c=4a^3/27-c$, an equation already solved by al-Ṭūsī in the first part of the *Treatise*. He next justifies this affine transformation. He uses also the affine transformation $x_1=y+a-x_2$, with y a positive solution of an equation solved earlier in the *Treatise*. Al-Ṭūsī justifies this last affine transformation and finally shows that $x_1 \neq x_0$ and $x_1 \neq x_2$.

In equation (4) there suddenly appeared a difficulty, since the maximum $f(x_0)$ could be negative. Al-Ṭūsī then imposes a necessary condition, to consider only the case where $f(x_0)>0$, and next proceeds as before. The equation $f'(x)=0$ has then two roots x_0' and x_0 $(x_0' < x_0)$, which correspond respectively to a negative minimum and a positive maximum. Al-Ṭūsī only considers the root x_0 and obtains $c_0=f(x_0)$. However, the equation $f(x)=0$ has, in this case, three roots, 0, $\lambda_1>0$, $\lambda_2>0$, with $\lambda_1<\lambda_2$. Al-Ṭūsī deduces that, for $c<c_0$, equation (4) has two positive roots x_1 and x_2 such that

$$0<\lambda_1<x_1<x_0<x_2<\lambda_2$$

This quick summary shows that the presence of the idea of the derivative is neither fortuitous nor secondary but, rather, intentional. It is true, however, that this is not the first time that one encounters the expression of a derivative in the *Treatise:* it is already introduced by al-Ṭūsī to construct a numerical method of solution of equations. This method goes as follows: al-Ṭūsī determines the first decimal digit of the root, as well as its decimal order. The root is then written as $x=s_0+y$, with $s_0=\sigma_0\times10^r$ (*r* the decimal order). He determines next the second digit with the help of the equation in y, $f(s_0+y)=0$; this algorithm, called Ruffini-Horner, is used to determine the different terms of the preceding cubic equation in *y*. The algorithm introduced by al-Ṭūsī serves to arrange the calculations so as to minimize the number of necessary multiplications, and is none other than a slightly modified form of the Ruffini-Horner algorithm adapted for cubic equations. Al-Ṭūsī then introduces as the coefficient of y the value $f'(s_0)$ of the derivative of f at s_0. Al-Ṭūsī obtains the last digit of *y,* i.e. the second digit of the required root, by taking the integer part of

$$-f(s_0)/f'(s_0)$$

We recognize here the method known as 'Newton's' for the approximate solution of equations. After having determined the second digit, which is the first of *y,* one applies the same algorithm to the equation in *y* to find a third digit, and one continues like this until the root is obtained, which is an integer in the cases considered by al-Ṭūsī.[31] However, if it were not, one finds the numbers after the decimal point by continuing as before. The successors of al-Ṭūsī proceeded in this way for the case where the root is not an integer, as explained in the text of al-Aṣfahānī, in the nineteenth century.[32]

If the presence of an expression for the derivative is not doubted, it remains that al-Ṭūsī did not explain the route which led him to such a notion. To understand better the originality of his method, we consider the example of equation (3) which is written

$$f(x)=x(b-ax-x^2)=c$$

The fundamental problem is to find the value $x=x_0$ at which the maximum is reached. Now it is in explaining the splitting of equation (3) into two equations solved beforehand by means of affine transformations

$$x\rightarrow y=x-x_0 \quad \text{and} \quad x\rightarrow y=x_0-x$$

that al-Ṭūsī gives

$$f(x_0) - f(x_0 + y) = 2x_0(x_0 + a)y - (b - x_0^2)y + (3x_0 + a)y^2 + y^3$$

and

$$f(x_0) - f(x_0 - y) = (b - x_0^2)y - 2x_0(x_0 + a)y + (3x_0 + a)y^2 - y^3$$

Al-Ṭūsī has to compare $f(x_0)$ with $f(x_0+y)$ and $f(x_0-y)$ noting that on $]0, \lambda_2[$ the terms

$$y^2(3x_0+a+y) \quad \text{and} \quad y^2(3x_0+a-y)$$

are positive. Next, he can deduce two equalities such that

$$\begin{array}{ll} \text{if } b - x_0^2 \geqslant 2x_0(x_0 + a) & \text{then } f(x_0) > f(x_0 + y) \\ \text{if } 2x_0(x_0 + a) \geqslant b - x_0^2 & \text{then } f(x_0) > f(x_0 - y) \end{array}$$

and in consequence

$$b - x_0^2 = 2x_0(x_0 + a) \Rightarrow \begin{cases} f(x_0) > f(x_0 + y) \\ f(x_0) > f(x_0 - y) \end{cases}$$

i.e. if x_0 is the positive root of the equation

$$f'(x) = b - 2ax - 3x^2 = 0$$

then $f(x_0)$ is the maximum of $f(x)$ in the interval studied. We notice that the two equalities correspond to the Taylor expansion with

$$f'(x_0) = b - 2ax_0 - 3x_0^2 \qquad \frac{1}{2!} f''(x_0) = -(3x_0 + a)$$

$$\frac{1}{3!} f'''(x_0) = -1$$

This method of al-Ṭūsī consists then, it seems, of arranging $f(x_0+y)$ and $f(x_0-y)$ according to powers of y, and of showing that there is a maximum when the coefficient of y is zero in this expansion. The value of x for which $f(x)$ is maximum is therefore the positive root of the equation represented by $f'(x)=0$. The virtue of the affine transformations $x \rightarrow x_0 \pm y$, with x_0 the root of $f'(x)=0$, is that the term in y in the new equation vanishes. It is probable that starting from this property that al-Ṭūsī discovered the derivative equation $f'(x)=0$, perhaps together with consideration of the graph representing f which he never draws in the *Treatise*. For small y, the principal part of the variation of $f(x_0 \pm y)$ is in y^2 and does not change sign with y. I have shown elsewhere that the method of al-Ṭūsī resembles strongly that of Fermat, in the latter's investigation of maxima and minima of polynomials.[33]

As we have just seen, the theory of equations is no longer only an area of algebra but covers a much wider domain, The mathematician gathers within this theory the geometrical study of equations and their numerical solution. He poses and solves the problem of the possible conditions for each equation, which leads him to devise the local study of curves that he uses, and notably to study systematically the maximum of a third degree polynomial by means of the derivative equation. In the course of the numerical solution, he does not only apply certain algorithms where one meets again the idea of the derivative of a polynomial, but he tries hard to justify these algorithms with the help of the idea of 'dominant polynomials'. It is clear that it is a mathematics of a very high level for this epoch; put simply, here already we touch the limits of a mathematical research carried out without efficient symbolism. All the research of al-Ṭūsī was done in fact in natural language, without any symbolism (except, perhaps, for a certain symbolic use in tables), which made it particularly complicated. One such difficulty appears as an obstacle, not only to the internal progress of the research itself, but also to the communication of results. In other words, as soon as a mathematician handles analytical ideas, such as those mentioned above, natural language is found to be inadequate to express the concepts and operations which were applied, constituting a limit to innovation in, as well as to diffusion of, this mathematical knowledge. The followers of al-Ṭūsī were, in all likelihood, affected by this same obstacle, until mathematical notation was truly transformed, after Descartes especially.

But the example of al-Ṭūsī suffices to show that the theory of equations not only was transformed after al-Khayyām but did not cease to demarcate itself even more clearly from search for solutions by radicals; it thus ended up by covering a vast field, including sectors which later would belong to analytical geometry, or even to analysis.

But what was the destiny of this theory of equations of al-Ṭūsī? This question is still at the edges of research, and we cannot at this time provide a satisfactory answer. We do not know of work in algebra by his student Kamāl al-Dīn ibn Yūnus. However, the student Athīr al-Dīn al-Abharī (d. 1262) of Kamāl al-Dīn ibn Yūnus composed an algebra which reached us abridged, according to the copyist. But, in the part that we do have, he applies the method of numerical solution of al-Ṭūsī, and in the same terms, to the equation $x^3=a$. Al-Khilāṭī,[34] another algebraist of this time, recalls that al-Ṭūsī was 'the master of his master' and that he studied cubic equations, but he himself was faithful to the tradition of al-Karajī. Other evidence of this time mentions al-Ṭūsī,[35] but nothing has come down to us which indicates that one or the other of the mathematicians had taken up the theory of al-Ṭūsī. Whilst in effect we find traces of the book of al-Ṭūsī with his followers, we do not know for the moment of any commentaries on his algebra.

Such could have existed but, even if this were the case, we doubt that it could surpass the work of al-Ṭūsī without setting out the operative notation necessary to develop the analytical ideas already contained in al-Ṭūsī's *Treatise on Equations*.

NOTES

1 In the preamble of his book, al-Khwārizmī mentions the generous encouragement of the arts and sciences by the Caliph al-Ma'mūn, who had encouraged him to write his book. Now al-Ma'mūn reigned between 813 and 833, which consequently are the limits for the dating of the book. Cf. al-Khwārizmī, *Kitāb al-jabr*.

2 The title of the book is *Kitāb al-jabr wa al-muqābala*. Recall that the two terms *al-jabr* and *al-muqābala* refer to both a discipline and two operations. Consider for example

$x^2+c-bx=d$ with $c>d$

Al-jabr consists in transposing the subtractive expressions

$x^2+c=bx+d$

and *al-muqābala* in reducing to similar terms:

$x^2+(c-d)=bx$

3 Cf. al-Khwārizmī, *Kitāb al-jabr,* p. 16.

4 Thus, Abū Kāmil writes about al-Khwārizmī: 'The one who first achieved a book of algebra and of *al-muqābala;* the one which started and invented all the fundamentals found there'; Abū Kāmil, MS Kara Mustafa, 379, folio 2[r]. The same Abū Kāmil wrote: 'I have established, in my second book *(al-Waṣāya bi-al-jabr)* the proof of authority and priority in algebra and al-muqābala of Muḥammad ibn Mūsā al-Khwārizmī, and I replied to a hot-head called Ibn Barza, about what he attributed to 'Abd al-Ḥāmid, whom he mentioned was his grandfather.' Cf. Ḥajjī Khalīfa, vol. 2, pp. 1407–8. We can multiply the evidence which is abundant in this area. Sinān ibn al-Fatḥ, in the introduction to his pamphlet, only mentions al-Khwārizmī, confirming that algebra was his creation: 'Muḥammad ibn Mūsā al-Khwārizmī wrote a book which he called algebra and *al-muqābala*.

5 Cf. al-Khwārizmī, *Kitāb al-jabr,* pp. 20–1.

6 *Ibid.,* pp. 21–2.

7 *Ibid.,* p. 27.

8 Cf. Aydin Sayili, pp. 145 *et seq*.

9 Thābit ibn Qurra: *Fī taṣḥīḥ masā'il al-jabr bi-al-barāhīn al-handasiyya,* MS Topkapi Saray, Ahmet III, no. 2041, folio 245[r].

10 *Ibid.,* folio 246[v].

11 It is an anonymous manuscript (no. 5325 Astan Quds, Meshhed, folio 24^{r-v}) falsely attributed to Abū Kāmil; copied in 581 H/1185.

12 See later, note 24.

13 On powers in Sinān ibn al-Fatḥ, see Rashed (1984:21 n.11).

14 Abū Kāmil, see note 4.

15 Let $a+\sqrt{b}$ be first binomial, i.e.

$$a \in Q \qquad b \in Q \qquad a > \sqrt{b} \qquad \sqrt{b} \notin Q \qquad \frac{(a^2-b)^{1/2}}{a} \in Q$$

Then $a-\sqrt{b}$ is a first apotome.

16 Al-Māhānī, *Tafsīr al-maqāla al-ʿāshira min kitāb Uqlīdis,* MS BN Paris 2457, folios 180^{v}–187^{r} (cf. especially folio 182^{r}).

17 Cf. Diophantus, *Les Arithmétiques*.

18 Abū al-Wafāʾ al-Būzjānī*: Fī jamʿ aḍlāʿ al-murabbaʿāt wa al-mukaʿʿ wa akhdh tafāḍulahā,* MS 5521 Astan Quds, Meshhed.

19 This is what al-Samawʾal writes, after having noted in a table, on either side of x^0, the powers: 'If the two powers are on either side of unity, from one of them we count in the direction of unity the number of elements in the table which separate the other power from unity, and the number is on the side of unity. If the two powers are on the same side of unity, we count in the opposite direction to unity' (al-Samawʾal, French Introduction, p. 19).

20 *Ibid.,* p. 37.

21 Ibn al-Bannāʾ: *Kitāb fī al-jabr wa al-muqābala,* MS Dar al-Kutub, Riyāḍa, M., folio 1.

22 Al-Sulamī: *Al-muqaddima al-kāfiya fī ḥisāb al-jabr wa al-muqābala,* Collection Paul Sbath, no. 5, folios 92^{v}–93^{r}.

23 Cf. van Egmond, (1983).

24 Ibn al-Bannāʾ, *Kitāb fī al-Jabr,* folio 26^{v}.

25 Compare the manuscript falsely attributed to Abū Kāmil, n. 11, folio 25.

26 Here is how al-Khayyām recounts this history in his own way in his celebrated treatise on algebra: 'As for the Ancients, nothing has come down to us of what they said: perhaps, after having researched and examined them, they had not grasped them; perhaps their research had not obliged them to examine; perhaps finally nothing of which they said has been translated into our language. As for the Moderns, it is al-Māhānī Abū ʿAbdallāh Muḥammad b. ʿĪsā Aḥmad al-Māhānī, he lived between about 825 and 888] who amongst them has led to the algebraic analysis of the lemma that Archimedes used, considering it as admitted, in proposition 4 of the second book of his work on *The Sphere and the Cylinder;* now he has arrived at cubes, squares and numbers forming an equation that he does not succeed in solving even after much thought; he thus ended by judging that it was impossible, until Abū Jaʿfar al-Khāzin appeared and solved the equation by conic sections.

Following him, some geometricians had need of several types of [these equations], and certain of them solved some; but none of them said anything definite about the enumeration of their types, nor about [the means] of obtaining the forms of each of them, nor about their demonstrations, except for the two types that I will mention.

As for me, I have wished, and still ardently do, to know with certitude all these types, and to distinguish, amongst the forms of each of them, the possible cases from the impossible cases, through demonstrations; I know in effect that one has a very urgent need when one is wrestling with the difficulties of the problems. However, I have not been able to dedicate myself exclusively to the acquisition of this, nor to think about it with perseverance, distracted as I have been by vicissitudes. For we find ourselves tried by the dwindling of the men of science, with the exception of a group as small as its afflictions are large, and whose worry is to find time on the wing to dedicate to the achievement and the perfectioning of the science'. This text is fundamental for the history of cubic equations. See our edited and translated version, with commentary, of *L'Oeuvre Algébrique d'al-Khayyām,* pp. 11–12.

27 *Ibid.,* pp. 82–4: 'As for the ancient mathematicians, who did not speak in our language, they had attracted attention to nothing of all of this, or nothing has reached us which has been translated into our language.

And among the Moderns, who speak our language, the first who had need of a trinomial sort of these fourteen kinds is al-Māhānī, the geometrician. He solved the lemma which Archimedes has taken, considering it as admitted, in proposition 4 of the second book of his work on *The Sphere and the Cylinder*. It is this which I am going to explain.

Archimedes said: the two straight lines AB and BC are of known magnitude, and one is in the prolongation of the other; and the ratio of BC to CE is known. CE is therefore known, as is shown in the *Data* [of Euclid]. He then said: let us set the ratio of CD to CE equal to the ratio of the square of AB to the square of AD.

He did not say how this was known, since one had necessarily to have conic sections. And, besides this, he introduced nothing in the book which was founded on conic sections. He also took this as admitted. The fourth proposition concerns the division of a sphere by a plane, according to a given ratio. But al-Māhānī used the terms of algebraists in order to facilitate [the construction]; as the analysis led to numbers, squares and cubes in equations, and as he could not solve them by conic sections, he thus ended by saying that it is impossible. The solution of one of these types therefore remained hidden from this eminent man, in spite of his eminence and his primacy in this art, until Abū Jaʿfar al-Khāzin appeared and indicated a method which he described in his treatise; and Abū Naṣr b. ʿIrāq, protégé of the Prince of Believers from the land of Khwārizm, solved the lemma that Archimedes had assumed to determine the side of a heptagon inscribed in a circle, and which is founded on the square verifying the mentioned property: he used algebraic terms. The analysis led to [the equation] "a cube plus squares equal a number", which he solved by sections.

This man, by my life, is of an excellent class in mathematics. This is the problem in the face of which Abū Sahl al-Qūhī, Abū al-Wafāʾ al-Būzjānī, Abū Ḥāmid al-Ṣāghānī, and a group of their colleagues, who were all devoted to His Lordship ʿAḍud al-Dawla, in the City of Peace [Baghdad], were found to be powerless; the problem, I say, is as follows: if you divide ten into two parts, the sum of their squares plus the quotient of the largest over the smallest is seventy-two. Analysis leads to squares equal to the roots plus a number. These eminent men were totally perplexed for a long time when faced with this problem, until Abū al-Jūd solved it. They have conserved [his solution] in the library of the Samanid kings. There are thus three kinds of compound equations, two trinomials and a quadrinomial. The only binomial equation, i.e. "the cube is equal to a number", our eminent predecessors solved. Nothing from them has reached us about the ten [equations] which remain, nor anything as detailed. If time granted us and if success accompanies me, I will record these fourteen types with all their branches and sections, distinguishing between the possible and impossible cases—in fact certain of these types require some conditions for them to be valid—in a treatise which will contain many of the lemmas preceding them, of great utility for the principles of this art.'

28 *ibid.,* p. 80.

29 See Rashed (1974b).

30 Cf. Sharaf al-Dīn al-Ṭūsī.

31 Taking the example of the numerical resolution of the equation

$$x^3=bx+N$$

al-Ṭūsī writes:

> To determine the required number, we place the number in the table and we count its rows by cubic root, no cubic root, cubic root. We place the zeros of the cubic root, we count also [the rows] of the number by root, no root, until we arrive at the homonymous root of the last place assigned a cubic root. We next place the number of roots, and we count the rows by root, no root. The homonymous row of the last place assigned a root for this number of roots is the last row of the root of the number of roots. The problem has two cases.
>
> *First case:* The homonymous root in the last place assigned a cubic root is much greater than [the row] of the last part of the number of roots, as when we say: a number of the form 3 2 7 6 7 0 3 8 plus nine hundred and sixty-three roots equals a cube. We count from the homonymous root of the last place assigned a cubic root until the last row of the number of roots, and we count the same number from the last place assigned a cubic root in this direction; and there where we end up, we place the last part of the number of roots reduced to one third; we then have this figure:
>
> 3 2 7 6 7 0 3 8
> 3 2 1
>
> Since the homonymous root of the last place assigned a cubic root is the third place assigned a root, it is in the row of tens of thousands which is higher than the last row of the number of roots, which is in [the row] of hundreds. We count from the row of the homonymous root of the last place assigned a cubic

root until the hundreds, and we count through this number also from the row of the last place assigned a cubic root, finishing in the tens of thousands; we place the last part of the third of the number of roots in this row and we place next the required cubic root, which is three, at the place of the last zero. We subtract its cube from what is beneath it, we multiply it by the rows of a third of the number of roots and we add three times the product to the number. We put the square of the required number parallel to itself under the number, according to this figure:

```
3
6 0 5 5 9 3 8
    3 2 1
9
```

We subtract the third of the number of roots from the square of the required number and we remove the third of the number of roots; there remains then this figure:

```
3
6 0 5 5 9 3 8
8 9 6 7 9
```

We move the upper line by two rows and the lower line by one row; we place the second required number, two, and we subtract its cube from the number; we multiply it by the first required number, we add the product to the lower line, we multiply it by the lower line and we subtract three times each product of the number; we add the square of the second required number to the lower line, we multiply it by the first required number, we add the product to the lower line and we move the upper line by two rows and the lower line by one row. We place another required number, which is one; we subtract its cube from the number, we multiply it by the first required number and the second, we add the result to the lower line, we multiply it by the lower line and we subtract three times this product from the number. The upper line is then the figure 3 2 1 which is the required root.

Second case: The last row of the number of roots is greater than the homonymous root of the last place assigned a cubic root, as when we say: a number of roots equal to 1 02021 plus a number of the form 3 2 7 4 2 0 equal a cube. We count the number of roots by root, no root, and we add to the number two rows by putting zeros in front of it; we look for the place assigned the highest root corresponding to the number of roots; we then place the zeros of the cubic root and then we look for the highest homonymous cubic root of this place assigned a root. We move the row of the number of roots parallel to this root, so that it is parallel to the cubic root which is homonymous. We place the other rows of the number of roots in order; one has then this figure:

0 0 3 2 7 4 2 0
1 0 2 0 2 1

because the highest place assigned a root which corresponds to them is the third, and it is in the column of tens of thousands: its homonymous is the third place assigned a cubic root which is in the [column] of thousands of thousands. We move the row of tens of thousands of the number of roots so that it is parallel to the place assigned the third cubic root, and we look for the greatest number such that one can remove its square of the number of roots; it is three; we place it in the third place assigned a cubic root; we multiply it by the rows of the number of roots, we add the product to the number and we remove its cube of the number. We reduce the number of roots to a third; it will begin then at the row of hundreds according to this figure:

3
3 9 3 3 7 2 0
3 4 0 0 7

We place next the square of the required number parallel to it below the number; one subtracts the third of the number of roots from it and deletes the line which is the third of the number of roots; we move the upper line by two rows and the lower line by one row and we apply the procedure until it is finished.

(Sharaf al-Dīn al-Ṭūsī, vol. I, pp. 49–52; see Table VI, p. cvii and Table VII, p. cviii)

32 Sharaf al-Dīn al-Ṭūsī, vol. I, pp. 118 *et seq.* On the other hand, al-Aṣfahānī gives in the same treatise an interesting method for finding a positive root of a cubic equation, based on the property of a fixed point. Did he take it from his ancient predecessors, as he did for the method of al-Ṭūsī? He probably did, but at this time we cannot settle such a question. Here, described quickly, the method is applied to the same example of al-Aṣfahānī. Solve the equation

$$x^3 + 210 = 121x \quad \text{with} \quad x \in R_+$$

We write this equation in the form

$$x=(121x-210)^{1/3}=f(x)$$

Al-Aṣfahānī takes then $x_1' = 11$, whence

$$y_1 = f(x_1') = (1121)^{1/3} < 11$$

He takes an approximate value by default of y_1, namely 10.3; he finds

$f(10.3)=(1036.3)^{1/3}<10.3$

He takes then $x'_2 = 10.3$ and $y_2 = f(x'_2) = (1036.3)^{1/3}$. He takes next an approximate value by default of y_2, namely 10.1. He finds that

$f(10.1)=(1012.1)^{1/3}<10.1$

He takes then $x'_3 = 10.1$ and so on; the first terms of this series are

$$x'_1 = 11 > x'_2 = 10.3 > x'_3 = 10.1 > \ \dots$$

Note that al-Aṣfahānī chooses the value 11 in a manner that is a little different. Instead of the function *f,* he considers a function *g* such that $f \le g$, i.e.

$g(x)=(121x)^{1/3}$

and looks for a root x_1 of the new equation $x=g(x)$, which ensures that $x_1=11>x_0$ if x_0 is the required root.

33 Sharaf al-Dīn al-Ṭūsī, vol. I, p. xxvii.

34 Al-Khilāṭī, *Nūr al-Dalāla fī 'ilm al-jabr wa al-muqābala,* MS of the University of Teheran no. 4409, folio 2.

35 See Shams al-Dīn al-Mārdīnī, *Niṣāb al-ḥabr fī ḥisāb al-jabr,* Istanbul, MS Feyzullah no. 1366, folios 13–14.

12
Combinatorial analysis, numerical analysis, Diophantine analysis and number theory

ROSHDI RASHED

The successors of al-Khwārizmī not only set about the application of arithmetic to algebra, but they also applied algebra revived by al-Karajī to arithmetic, trigonometry and the Euclidean theory of numbers. These applications, like those of algebra to geometry and of geometry to algebra examined in the preceding chapter, always led to the founder acts of new mathematical disciplines, or at least, of new areas. Algebra, it cannot be emphasized enough, has in this way played a central role not only to restructure and organize the disciplines of Hellenistic heritage, in order to widen their domains and methods, but also and above all to create new ones. It is in this way that combinatorial analysis, numerical analysis, new elementary number theory, integer Diophantine analysis came about. It is the story of these areas, yesterday still for the most part unknown, that we are going briefly to recapture.[1]

COMBINATORIAL ANALYSIS

One thing is to search for combinatorial activity in an innocent way, i.e. where it appears without any particular intention, for example the combining of terms, such as number, thing, square, cube to enumerate all the forms of an equation of a degree less than or equal to 3; it is a completely different thing to seek it where one is attempting to elaborate all its rules and laws. Only these investigations amount to the constitution of combinatorial analysis as an area of mathematics. Now, this combinatorial activity started by appearing in such a way, but in a dispersed manner, among linguists on the one hand and algebraists on the other. It is only later that the liaison was made between the two currents, and that combinatorial analysis presented itself as a mathematical instrument applicable to the most diverse situations: linguistic, philosophical, mathematical etc. In the ninth century, one encounters this activity among linguists and philosophers who posed problems connected with language, in three notable areas: phonology, lexicography and finally cryptography. The name al-Khalīl ibn Aḥmad (718–86) signifies the history of these three disciplines. The latter resorted explicitly, for the constituting of Arabic lexicography, to the calculation of arrangements and combinations. In effect in his book, *Kitāb al-ʿAyn*, al-Khalīl[2] tries to rationalize the empirical practice of lexicography. He therefore wants to achieve an enumeration in an exhaustive way of the words of the language, and, on the other hand, to find a means by which there is a one-to-one relation between the set of the words and the entries of lexis. Al-Khalīl elaborates then the theory here: the language is a phonetically realized part of the possible language. If in effect the arrangement r to r of the letters of the alphabet, with $1<r\leq 5$, according to the number of letters of the root in Arabic, gives us, says al-Khalīl, the ensemble of the roots, and consequently of the words, of the possible language, one single part, limited by the rules

of incompatibility of the phonemes of roots, will form the language. To compose a lexis going back therefore to constituting the possible language by means of extracting next, according to the stated rules, all the words which submit to it—an important thesis, the formula of which has, however, required a phonological study of Arabic, which al-Khalīl first undertook. For the composition of the lexis, al-Khalīl starts by calculating the number of combinations—without repetition—of the letters of the alphabet, taken r to r with $r=2,\ldots, 5$, and then the number of permutations of each group of r letters. In other words, he calculates

$$A_n^r = r!\binom{n}{r}$$

n being the number of letters of the alphabet, $1<r\leq 5$.

Now this theory and the calculation by al-Khalīl are found in the writings of the majority of lexicographers. They will serve in other respects in cryptography, developed from the ninth century by al-Kindī, and then, at the end of the same century and the beginning of the following, by linguists such as Ibn Waḥshiyya, Ibn Ṭabāṭabā, among many others. In the practice of their discipline, the cryptographers resorted to the phonological analysis of al-Khalīl, for the calculation of the frequency of letters in Arabic and the calculation of the permutations, the substitutions and the combinations. A good number of great linguists, starting with al-Khalīl himself, have left us writings on cryptography and cryptoanalysis.[3]

At the same time as this important combinatorial activity, the algebraists, as we have seen, had expressed and proved, at the end of the tenth century, the rule for formation of the arithmetic triangle for the calculation of binomial coefficients. Al-Karajī[4] had in effect given the rule

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

and the development

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r} a^{n-r} b^r$$

The algebraists were applying the new rules in their calculations. Al- Samaw'al for example[5] gives himself ten unknowns, and looks for a system of linear equations of six unknowns. He then combines the ten decimal figures considered so to speak as symbols of these unknowns—one would today say their indices—six to six, and obtains thus his system of 210 equations. He proceeds also through combinations to find the 504 conditions of compatibility of this system. All these combinatorial activities, these rules discovered in the course of linguistic research and algebraic studies, have constituted the concrete conditions of the emergence of this new area of mathematics. It remains, however, that the act of the birth of the latter resides in the explicitly combinatorial interpretation of the arithmetic triangle, of its law of formation etc., i.e. the rules given by al-Karajī as instruments for calculation. It would be excessive to think that the algebraists would not have seized this interpretation quite soon. On the contrary we are more and more convinced that this interpretation had been seen by the algebraists, but that nothing was imposing upon them to give an explicit formulation. Such a necessity was felt when the rules of combinatorial calculation were starting to be applied for dealing with problems as much mathematical as non-mathematical, but that were wanting to be solved mathematically. The example of al-Samaw'al testifies in some way to this fact; the combinatorial interpretation is obviously there, very probably before the thirteenth century, as we are now in a position to show thanks to a text hitherto unknown, by the mathematician and philosopher Naṣīr al-Dīn al-Ṭūsī (1201–73). The reading of this text[6] shows that he knew this interpretation, and presents it totally naturally as an accepted thing and expresses it in terminology that one will find, totally or in part, among his successors. In this writing, al-Ṭūsī wants to respond to the following metaphysical question: 'How do an infinity of things come from the first and unique principle?', i.e. how to explain the infinity starting from the One? We do not intend here to examine the metaphysical question of al-Ṭūsī, but only to recall the intention: to solve this philosophical problem mathematically. In the course of this solution, he was brought to calculate the number of combinations of n distinct objects taken k to k, with $1\leq k\leq n$. Thus, he calculates for $n=12$

$$\sum_{k=1}^{n}\binom{n}{k}$$

and uses in the course of his calculation the equality

$$\binom{n}{k}=\binom{n}{n-k}$$

Note now that al-Ṭūsī had given in his book of arithmetic[7] the arithmetic triangle, its law of formation etc. Here, he applies certain of these rules. But, in order to interpret this calculation, al-Ṭūsī then takes twelve letters of the alphabet, and combines them in order to deduce his formulae.

Al-Ṭūsī reverts next to his initial problem, and considers, besides $n=12$ elements, $m=4$ basic elements, starting from which he had obtained the latter. The problem reverts in fact to considering two classes of objects, the first of $n=12$ different elements, the second of $m=4$ different elements, and to calculating the number of combinations that it is possible to make. Al-Ṭūsī calculates an expression equivalent to

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{p-k}\qquad 0\leqslant p\leqslant 16$$

whose value is the binomial coefficient

$$\binom{m+n}{p}$$

Starting from al-Ṭūsī at least, and very probably before him, one will not cease to find the combinatorial interpretation of the arithmetic triangle and of its law of formation, as the ensemble of the elementary rules of combinatorial analysis. As we have shown, towards the end of the same century and at the start of the fourteenth century, Kamāl al-Dīn al Fārisī (d. 1319), in a paper on the theory of numbers, reverts to this interpretation and establishes the usage of the arithmetic triangle to numerical orders, i.e. the result that one originally attributes to Pascal. In effect, to form the figurate numbers,[8] al-Fārisī establishes a relationship equivalent to

$$F_p^q = \sum_{k=1}^{p} F_k^{q-1} = \binom{p+q-1}{q}$$

with F_p^q the *p*th figurate number of order *q*. $F_1^q = 1$ for all *q*.

But, while al-Fārisī was devoting himself to these works in Iran, Ibn al- Bannā'[9] (d. 1321) was busying himself at the same time in Morocco with combinatorial analysis. The latter in effect reverts to combinatorial interpretation, and takes up the rules known before him, notably those of arrangement of *n* different objects, without repetition, *r* to *r*, of permutations, of combinations without repetition:

$$(n)_r = n(n-1)\ \ldots\ (n-r+1)$$

$$(n)_n = n!$$

$$\binom{n}{r} = \frac{(n)_r}{r!}$$

relationships easily deducible from the expression (*) given by al-Karajī three centuries previously.

Al-Fārisī and Ibn al-Bannā' not only succeed al-Ṭūsī, but they use the major part of lexis already adopted by the latter. This community of interest, and, for the most part, of vocabulary, shows well that it is about a tradition, and confirms a conjecture supported by us[10] ten years ago, i.e. that combinatorial analysis was well constituted as an area before al-Fārisī and Ibn al-Bannā'. With these authors, combinatorial analysis no longer has as its domain of application algebra or linguistics only, but more varied domains, metaphysics for example, i.e. every domain where one is dealing with the partition of a set of objects.

This conception and this subject area survive to this day. One will continue to treat combinatorial analysis in the different works of mathematics, and one will devote independent treatises to it. Thus, the subsequent mathematicians like al-Kāshī,[11] Ibn al-Malik al-Dimashqī,[12] al-Yazdī,[13] Taqī al-Dīn ibn Maʿrūf, to name but a few, deal with it. The first three take up the arithmetic triangle, its rules and its applications, the last takes up the example of the linguistic derivation in his *Arithmetics,*[14] to give the formula for permutations. As regards those who have composed independent treatises, we recall here for the first time al-Ḥalabī, who takes up the group of elementary formulae, the preceding text of al-Ṭūsī in a relatively long commentary, and who brings in a theoretical explanation to distinguish between arrangements with or without repetition, taking order into account or not; he undertakes the same task for combinations, and does not hesitate in involving himself in long calculations for the time.[15] In order to facilitate these calculations, he presents that which is already implicit in the writing of al-Ṭūsī: the relationship between figurate numbers and the number of different combinations, thanks to Table 12.1, for n=12.

Table 12.1

p \ q	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10	11	12
3	1	3	6	10	15	21	28	36	45	55	66	
4	1	4	10	20	35	56	84	120	165	220		
5	1	5	15	35	70	126	210	330	495			
6	1	6	21	56	126	252	462	792				
7	1	7	28	84	210	462	924					
8	1	8	36	120	330	792		5				
9	1	9	45	165	495		9					
10	1	10	55	220		0						
11	1	11	66		4							
12	1	12										
	1											

NUMERICAL ANALYSIS

Compared with Hellenistic mathematics, Arabic mathematics offers a much more important number of numerical algorithms. This trait is itself essential to the majority of historians, above all since the works of Luckey[16] on the fifteenth-century mathematician al-Kāshī. The relatively late dates of al-Kāshī, however, made it difficult to elucidate the true reasons of this character, to allow him to be put into historical perspective. This situation was profoundly altered when we were able to establish that the contribution of al-Kāshī came from a long way back, the twelfth century at least, as witnessed in the writings of al-Samaw'al[17] and Sharaf al-Dīn al-Ṭūsī.[18] The latter works to which we have recently added a writing by the mathematician and astronomer of the eleventh century, al-Bīrūnī, take us back several centuries earlier, and explain the reasons for the development of numerical techniques. These are intimately linked to algebra and to the astronomy of observation.

Algebra has not only supplied the theoretical methods indispensable to its development —which would not only be the study of polynomial expressions and combinatorial rules—but also a vast domain of the application of these techniques: the methods developed to determine the positive roots of numerical equations. The research in astronomy, on the other hand, encouraged mathematicians to take up again the problems of interpolation of certain trigonometric functions. Certain of these methods, it will be seen, have been applied in quantitative research in optics. The result, already worked out, is an appreciable number of numerical techniques that it is impossible to describe in so limited a space.

More important still than the number of numerical algorithms found by these mathematicians was the discovery of new axes of research like the mathematical justification of algorithms, the comparison between different algorithms with a view to choosing the best and, above all, the conscious reflections on the nature and the limit of the approximations.

It remains to return to the principal domains that were dividing numerical analysis: the extraction of roots of an integer and the numerical solution of equations, on the one hand, and the methods of interpolation, on the other.

The extraction of square and cube roots

As far as one goes back into the history of Arab mathematics, one encounters algorithms for the extraction of square and cube roots, of which some are of Hellenistic origin, while others are very probably of Indian origin and others lastly are due to Arab mathematicians themselves. But whatever the origins, near or remote, of these algorithms, the latter have been integrated in another mathematics which has given them a new extension in altering their meaning. It is thus that starting from the ninth century until the seventeenth century at least, every book of decimal arithmetic—i.e. every book of *ḥisāb*—or of algebra contains a statement on the extraction of square and cube roots, and sometimes, more generally, on the extraction of the *n*th root of an integer. If we emphasize these facts, it is to warn against the temptation to favour certain works, such as those of Kūshyār, al-Nasawī or Ibn al-Ḥaṣṣār. This advantage which is accorded to them is in effect by pure coincidence, and if their names are advanced into the works of historians, it is for the simple reason that their texts have been translated into a European language. Our first task will therefore be to retrace, at least in the outstanding points, the tradition from which these works were raised, which are not, however, the most advanced nor the most profound. Certain manuscript texts discovered by us will be of great help to us in this enterprise.

Let us start with al-Khwārizmī. In a book on arithmetic today lost[19] he proposed, according to what we are taught by the mathematician al-Baghdādī (d. 1037), a formula for approximating the square root of an integer *N*. If we write $N=a^2+r$ this formula is written, for *a* an integer

$$\sqrt{N}=a+\frac{r}{2a} \tag{1}$$

Al-Baghdādī does not omit to remind us that it is there a matter of an approximation by excess, which is far from being satisfactory—it suffices to be convinced to search for √2 and √3.[20]

But in the same period as al-Khwārizmī, the Banū Mūsā gave in their *Book on the Measurement of Plane and Spherical Figures*[21] another expression which will later be called 'the rule of zeros', and generalized it without difficulty for the extraction of the *n*th root. It is the expression

$$\sqrt[n]{N}=\frac{1}{m^k}\sqrt[n]{(Nm^{nk})} \tag{2}$$

with *m* and *k* two integers.

If one puts $m=60$ and $n=3$, one has the expression by the Banū Mūsā. This rule is found in the majority of books on arithmetic. Thus, to take but three examples, one finds it in *al-Fuṣūl,* compiled by al-Uqlīdisī in 952 for the square and cube roots,[22] in *al-Takmila* by al-Baghdādī for the cube root,[23] in the *Treatise on Indian Arithmetic* by al Samaw'al (1172–3) for the *n*th root.

All indicate then that the mathematicians wanted to find the best formulae of approximation. Thus, al-Uqlīdisī in the previously cited treatise gives, among other expressions,

$$\sqrt{N}=a+\frac{r}{2a+1} \tag{3}$$

which will later be called the 'conventional approximation', and $2a+1$ 'conventional denominator', according to the terms of Naṣīr al-Dīn al Ṭūsī and later al-Kāshī.

Al-Baghdādī gives the 'conventional approximation' for the cube root of *N:* if we write $N=a^3+r$, *a* an integer,

$$\sqrt[3]{N}=a+\frac{r}{3a^2+3a+1} \tag{4}$$

In order not to lose ourselves in details, we shall not enumerate here the masses of formulae given by different mathematicians for the approximations of these roots. We shall stop, however, on two contributions from the end of the tenth century, which, without being in any way the same, are nevertheless linked, because it is about the algorithm that will lead to that of Ruffini-Horner. Kūshyār ibn al-Labbān applies this algorithm, of an Indian origin in all likelihood, in his *Arithmetics.*[24] We know at present that Ibn al-Haytham not only knew this algorithm, but that he endeavoured to give it a mathematical justification. It is his general process that we show here, but in a different language.

Let the polynomial $f(x)$ have with integer coefficients and the equation

$$f(x)=N \tag{5}$$

Let s be a positive root of this equation, and suppose that S_i (with $i\geq 0$) is a series of positive integers such that the partial sums

$$\sum_{i=0}^{k} s_i \leqslant s$$

One says that the s_i are the parts of s.

It is evident that the equation

$$f_0(x)=f(x+s_0)-f(s_0)=N-f(s_0)=N_0 \tag{6}$$

has for roots those of equation (5) reduced by s_0.

For $i>0$, we form by recurrence the equation

$$\begin{aligned} &f_i(x)=f(x+s_0+\ldots+s_i)-f(s_0+\ldots+s_i)\\ &=[N-f(s_0+\ldots+s_{i-1})]-[f(s_0+\ldots+s_i)]\\ &-f(s_0+\ldots+s_{i-1})]=N_i \end{aligned} \tag{7}$$

Thus, for example, for $i=1$ we have

$$\begin{aligned} &f_1(x)=f(x+s_0+s_1)-f(s_0+s_1)\\ &=[N-f(s_0)]-[f(s_0+s_1)-f(s_0)]\\ &=N_0-[f(s_0+s_1)-f(s_0)]=N_1 \end{aligned}$$

The method applied by Ibn al-Haytham, and justified by him, which is used by Kūshyār and which is called Ruffini-Horner, forms an algorithm which allows the obtaining of the coefficients of the ith equation taken from the coefficients of the $(i-1)$th equation. That is the principal idea of this method.[25]

Let us start with the extraction of the nth root that had already been found in the twelfth century, if not before. We have

$$f(x)=x^n$$

If one knows the binomial formula, given, we have noted, in the tenth century by al-Karajī, there is no need to know the Horner table. The coefficients of the ith equation will be in this case

$$\binom{n}{k}(s_0+\cdots+s_{i-1})^{n-k} \qquad k=1,\ldots,n \tag{8}$$

and

$$N_i=N_{i-1}-\sum_{k=1}^{n}\binom{n}{k}(s_0+\cdots+s_{i-1})^{n-k}s_i^k$$

After this preliminary, we return to Ibn al-Haytham and to Kūshyār, for the square and cube roots. Let

$$f(x)=x^2=N$$

One has then two cases.

First case: N is the square of an integer. Assume the root is of the form

$$s=s_0+\ldots+s_h$$

with

$$s_i=\sigma_i 10^{h-i} \qquad 0\le i\le h$$

The task of the mathematicians of the eleventh century is first to determine h and the numerals a,. Formulae (8) are rewritten

$$2(s_0+\cdots+s_{i-1}),\ 1, \qquad N_i=N_{i-1}-[2(s_0+\cdots+s_{i-1})s_i+s_i^2]$$

One then determines σ0 by the inequalities

$$\sigma_0^2 10^{2h} \leqslant N < (\sigma_0+1)^{2h}\ 10^{2h}$$

and $\sigma_1,\ldots,\sigma_k$ by

$$\sigma_i = \frac{N_i}{2(s_0+\cdots+s_{i-1}).10^{h-i}}$$

In these expressions the N_i, for $1\leq i\leq h$, are calculated starting from N_{i-1} by subtracting from it

$$[2(s_0+\cdots+s_{i-1})s_i+s_i^2]$$

For $i=h$, one obtains $N_h=0$.

Second case: N is not the square of an integer. Ibn al-Haytham uses the same method to determine the integer part of the root, and then gives as a formula of approximation that of al-Khwārizmī and that of the 'conventional approximation', which are rewritten respectively with these notations

$$(s_0+\cdots+s_h)+\frac{N_h}{2(s_0+\cdots+s_h)}$$

and

$$(s_0+\cdots+s_h)+\frac{N_h}{2(s_0+\cdots+s_h)+1}$$

Thus, he does not only describe the algorithm, like Kūshyār, but he endeavours to give mathematical reasons, by justifying the fact that these two approximations encompass the root.

To extract the cubic root of an integer the process is analogous. Let

$$f(x)=x^3=N$$

Here also one has two cases.

First case: N is the cube of an integer. In this case S_0 is determined in such a way that $s_0^3 < N$. Ibn al-Haytham, like his contemporaries, takes then $s_1=s_2=\ldots=s_h=1$.

The coefficients of the ith equation are rewritten

$$3(s_0+i)^2,\ 3(s_0+i),\ 1,\ N_i=N_{i-1}-[3(s_0+(i-1))^2+3(s_0+(i-1))+1]$$

If N_i is the cube of an integer, there exists a value of *i* such that $N_k=0$; i.e. such that s_0+k is the root sought. Just as his contemporaries, Ibn al-Haytham describes with all the details the different steps of the algorithm.

Second case: N is not the cube of an integer. Ibn al-Haytham also gives two formulae symmetrical to the two already mentioned for the extraction of the square root, and which are rewritten

$$(s_0+\cdots+s_h)+\frac{N_h}{3(s_0+\cdots+s_h)^2}$$

and

$$(s_0+\cdots+s_h)+\frac{N_h}{3(s_0+\cdots+s_h)^2+3(s_0+\cdots+s_h)+1}$$

One recognizes in the latter the 'conventional approximation'.

The ensemble of the methods and the preceding results, acquired at the start of the eleventh century, is found not only among the contemporaries of these mathematicians but in the majority of treatises on arithmetic, then very numerous. Amongst many others, let us recall the treatises of al-Nasawī,[26] successor of Kūshyār, of Naṣīr al-Dīn al-Ṭūsī,[27] Ibn al-Khawwām,[28] al-Baghdādī, Kamāl al-Dīn al-Fārisī[29] etc.

The extraction of the nth root of an integer

In possession of the arithmetical triangle and the binomial formula from the end of the tenth century, the mathematicians did not encounter many more major difficulties in the generalization of the preceding methods and for the formulation of the algorithm in the case of the *nth* root. And in fact such endeavours, unfortunately lost, already existed in the eleventh century with al-Bīrūnī and al-Khayyām; the ancient bibliographies including the titles of their treatises dedicated to this research are a testimony to it, but these vestiges tell us nothing of their methods. It is in his contribution of 1172–3 that al-Samaw'al[30] not only applies the method called Ruffini-Horner for the extraction of the *n*th root of a sexagesimal integer, but formulates a clear concept of approximation. By 'to approach', the mathematician of the twelfth century understood: to know a real number by means of a series of known numbers with an approximation that the mathematician can make as small as he wants. It is therefore a matter of measuring the distance between the *n*th irrational root and a series of rational numbers. After having defined the concept of approximation, al-Samaw'al starts by applying the method called Ruffini-Horner for the example

$$f(x)=x^5-Q=0$$

with $Q=0; 0, 0, 2, 33, 43, 3, 43, 36, 48, 8, 16, 52, 30$.

Now this method survived the twelfth century and is found in plenty of other treatises of 'Indian arithmetic' as one then said. One finds it still later among the predecessors of al-Kāshī, by al-Kāshī himself just as among his successors. To take only the example of the latter, in his *Key of Arithmetic* he solves

$$f(x)=x^5-N=0$$

with N=44 240 899 506 197.

That is, it is a method well known and diffused since the twelfth century at least among the Arab mathematicians. This is not, however, the only one. There are many others, all based on the binomial formula, without necessary recourse to the Horner algorithm. We want again to insist on the multiplicity and the diffusion of these methods which were circulating not only in the basic treatises on arithmetic, but also in those of commentators and mathematicians of a lower class. One single example suffices therefore, taken at random from authors never previously studied. It is by a commentator who lived before 1241 and the text concerns the book of a mathematician from Kairouan, himself of the second class. The first is al-Aḥdab of Kairouan, and the second Abū al-Majd ibn ʻAṭiyya,[31] who established a method for extracting the *n*th root, to prove it and give numerical examples of it. He thus gives the example of the fifth root of N=4 678 757 435 232. Ibn ʻAṭiyya assumes that the root is in the form $a+b+c$, with $a=\alpha.10^2$, $b=\beta.10$. Here are the principal steps of his algorithm.

He first writes $N-a^5=N_1$ and then calculates

$$\sum_{k=1}^{5}\binom{5}{k}a^{5-k}$$

He then multiplies the terms of this expression respectively by b, b^2, b^3, b^4 and b^5 in order to obtain

$$\sum_{k=1}^{5}\binom{5}{k}a^{5-k}b^k$$

and calculates

$$N_2=N_1-\sum_{k=1}^{5}\binom{5}{k}a^{5-k}b^k$$

He then calculates

$$\sum_{k=1}^{5}\binom{5}{k}(a+b)^{5-k}$$

He multiplies these terms respectively by c, c^2, c^3, c^4 and c^5 in order to obtain

$$\sum_{k=1}^{5}\binom{5}{k}(a+b)^{5-k}c^k$$

and reaches

$$N_3=N_2-\sum_{k=1}^{5}\binom{5}{k}(a+b)^{5-k}c^k=0$$

If now we come to the extraction of the nth irrational root of an integer, we encounter an analogous situation. In his *Treatise on Arithmetic,* al- Samaw'al gives a rule for approaching by fractions the non-integer part of the irrational root of an integer. This process comes back to solving the numerical equation

$$x^n=N$$

He starts by looking for the largest integer x_0 such that $x_0^n \leqslant N$. There are two cases.

1 $x_0^n=N \Leftrightarrow x_0$ is the exact root looked for. We have seen that al-Samaw'al has a sure method for obtaining this result when it is possible.
2 $x_0^n<N \Leftrightarrow N^{1/n}$ is irrational. In this case he expresses as the first approximation

$$x'=x_0+\frac{N-x_0^n}{\left[\sum_{k=1}^{n-1}\binom{n}{k}x_0^{n-k}\right]+1} \qquad (1)$$

i.e.

$$x'=x_0+\frac{N-x_0^n}{(x_0+1)^n-x_0^n}$$

It is thus a generalization of what the mathematicians called the 'conventional approximation'.

This approximation by default is of the same nature as that which the Arabic predecessors of al-Samaw'al set out, but it is much more general. While the arithmeticians who had not integrated the results of al-Karajī were limiting the application of this method to powers of 2 and 3, the rule is here extended to any power, as we find later among so many mathematicians, such as Naṣīr al-Dīn al-Ṭūsī and al-Kāshī. Moreover, it is to improve these approximations that the decimal fractions were devised in an explicit manner, as the example of al-Samaw'al shows.[32]

The extraction of roots and the invention of decimal fractions

It has been seen previously[33] that in the middle of the tenth century al-Uqlīdisī reached an intuitive idea of decimal fractions, in the course of his study of the division by two of odd integers. He writes: 'the sharing into two halves of one, in each position, is five in front of it. It follows that if we divide into two halves an odd number, we have then the half of the one, five, in front of it. We mark the position of the units with a sign ‹'› over it so that it can be recognized.'[34] This result, appreciable without any doubt, accompanied by an easy principle of notation, does not, however, constitute a true theory of decimal fractions, nor an explicit survey of them. It simply forms for us an empirical rule for calculation in the case of division by two. It is necessary to wait for the algebraists of the school of al-Karajī to have the general and theoretical statement. The latter have all naturally felt the necessity for these fractions when they are looking to follow as far as is wanted the approximation of the *n*th irrational root of an integer. In order to invent these fractions, they have taken advantage of the algebra of polynomials, its rules and its methods of presentation. The first known statement of these fractions, given in 1172–3 by al-Samaw'al,[35] leaves no doubt about the algebraic methods, or about the aim and the expected applications. This statement follows directly in the book by al-Samaw'al, *al-Qiwāmī fī al-Ḥisāb al-Hindī*, after the chapter dedicated to the approximation of the *n*th root of an integer. Even the title of the chapter on decimal fractions is significant: 'On the subject of the posing of a unique principle by which one can determine all the operations of partition *(al-tafrīq)* which are division, the extraction of the square root, the extraction of one side for all powers, and the correction of all the fractions that appear in these operations, indefinitely'.[36] This 'unique principle' brought out by al-Samaw'al is no other than that already known in algebra and already explained by him in his book *al-Bāhir,* i.e. that, from one part and the other of x^0, one has an identical structure. It suffices therefore to substitute 10^0 for x^0, and to the other algebraic powers those of 10 to obtain decimal integers and fractions, or according to the writing of al-Samaw'al, 'given that the proportional positions starting with the position of the units [10^0] follow one another indefinitely according to the proportion of the tenth, one thus assumes, on the other side [of 10^0] the positions of the parts ‹from ten following one another) according to the same proportion, and the position of the units [10^0] like an intermediary between the positions of the integers of which the units move in the same way indefinitely and the positions of the parts indefinitely divisible.'[37] Al-Samaw'al pursues this explanation and gives a table that we transcribe by substituting for the verbal expressions, 10^n, and not noting all the positions:

$$\begin{matrix} 10^{13} & 10^{12} & \ldots & 10^{9} & \ldots & 10^{6} & \ldots & 10^{3} & \ldots & 10 \\ & 4 & & 3 & & 2 & & 1 & & \end{matrix}$$

$$\begin{matrix} 10^{0} & 10^{-1} & \ldots & 10^{-3} & \ldots & 10^{-6} & \ldots & 10^{-9} & \ldots & 10^{-12} & 10 \\ 0 & & & 1 & & 2 & & 3 & & 4 & \end{matrix}$$

In order to write the fractions, al-Samaw'al separates the integer part from the fractional part, one by noting the numbers of different positions, the other by noting the denominator

$$\begin{matrix} 10^{0} & 10^{-1} & 10^{-2} & 10^{-3} & 10^{-4} & 10^{-5} & 10^{-6} \\ 3 & 1 & 6 & 2 & 2 & 7 & 7 \end{matrix} \qquad \text{or} \qquad \begin{matrix} 3 \\ 162277 \\ 1000000 \end{matrix}$$

In the same algebraic tradition as that of al-Samaw'al, al-Kāshī (d. 1436–7) takes up again much later the theory of decimal fractions, and gives an account of a great theoretical and calculatory mastery. He insists on the analogy between the two sexagesimal and decimal systems, uses fractions not only to approach the only real algebraic numbers but also the number π to $1/10^{16}$ approximately. In addition, to our knowledge he is the first to give a name to these fractions: *al-kusūr al-ʿashāriyya*, 'the decimal fractions'.[38]

The decimal fractions outlive al-Kāshī in the writings of the astronomer and mathematician of the sixteenth century Maʿrūf,[39] and al-Yazdī.[40] Several signs suggest that they were transmitted to the West before the middle of the seventeenth century, and they are named in a Byzantine manuscript brought to Vienna in 1562 as the fractions 'of the Turks'.[41]

Methods of interpolation

Methods of interpolation have been applied for a long time already by astronomers. Neugebauer showed that in certain Babylonian texts relating to the rising and setting of Mercury, astronomers were already proceeding with linear interpolations[42] two centuries before our time. Ptolemy also had recourse to this linear interpolation for the tables of chords. That is, the Arabic astronomers and mathematicians were well acquainted with this interpolation at least thanks to Ptolemy, and they gave it the significant title: method of the astronomers. Let us assume that $x_{-1}<x<x_0$ and $d=x_0-x_{-1}=x_i-x_{i-1}$ for $i=-2, -1,\ldots, n$; then the linear interpolation is rewritten

$$y = y_{-1} + \left(\frac{x - x_{-1}}{d}\right) \Delta y_{-1} \qquad (1)$$

with Δ the first difference of order 1.

From the ninth century the astronomers were already looking for methods to form and use astronomical and trigonometric tables, and on this occasion came back to methods of interpolation to improve them. It is in this way that in the tenth century two mathematicians at least had proposed processes of interpolation of the second order: Ibn Yūnus and al-Khāzin. The first gave an expression equivalent to

$$y = y_{-1} + \left(\frac{x - x_{-1}}{d}\right)\left[\frac{1}{2}(\Delta y_{-1} + \Delta y_0) + \frac{1}{2}\left(\frac{x - x_{-1}}{d}\right)\Delta^2 y_{-1}\right] \qquad (2)$$

It is evident that it is a parabolic interpolation; the curve defined by (2) passes through the point (x_{-1}, y_{-1}).

As for al-Khāzin,[43] he also gave a parabolic interpolation, a variant of what will be found in the work of al-Kāshī five centuries later.

But the major event in the history of the methods of interpolation in Arabic was the translation of the *zīj* of Brahmagupta, the *Khaṇḍakhādyaka*, and the research of al-Bīrūnī in this area.

We have been able to show recently[44] that al-Bīrūnī was familiar with the book of Brahmagupta, as well as his method of quadratic interpolation, which is rewritten

$$y = y_0 + \left(\frac{x - x_0}{d}\right)\left[\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{1}{2}\left(\frac{x - x_0}{d}\right)\Delta^2 y_{-1}\right] \qquad (3)$$

Now, from the text by al-Bīrūnī, this method assumes that $x<x_0$ and leads to the formula

$$y = y_0 + \left(\frac{x_0 - x}{d}\right)\left[\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{1}{2}\left(\frac{x_0 - x}{d}\right)\Delta^2 y_{-1}\right]$$

Al-Bīrūnī adds also another method of Indian origin which seems to be unknown in the historical literature, and which he calls by its Indian name the *sankalt* method or, in other words, the monomial method; it is written

$$y = y_0 - \frac{(x_0 - x)(x_0 - x + 1)}{d(d + 1)}\Delta y_{-1} \qquad (4)$$

This method proceeds by calculation of the increases in x_i towards x_{i-1}. Al-Bīrūnī himself, in his celebrated *al-Qānūn al-Mas'ūdī*, gives another method of interpolation, which is written

$$y = y_{-1} + \left(\frac{x - x_{-1}}{d}\right)\left[\Delta y_{-2} + \left(\frac{x - x_{-1}}{d}\right)\Delta^2 y_{-2}\right] \qquad (5)$$

Note that the application of this formula requires for the calculation of Δy_{-2} and $\Delta^2 y_{-2}$ that

$$x_{-2} = (x_{-1} - d) \in \left]0, \frac{\pi}{2}\right[$$

i.e. that $x_{-1} > d$.

The plurality of these methods at the end of the tenth century posed a new problem for research: how to compare these different methods between each other, in order to choose the most effective one for the tabular function studied? Al-Bīrūnī himself started to ask this question, and to confront different methods for the case of the cotangent function, with its difficulties due to the existence of the poles. In the following century, **al-Samaw'al** yet more explicitly attacked this task. He endeavoured to improve the methods stated by al-Bīrūnī, or inherited by him from the Indians. **Al-Samaw'al** started with the idea of weighting, and proposed the use of weighted means, taking into account the relative importance of Δy_{i-1} and Δy_i. Now it is this examination of the comparative performance of the methods which led the mathematicians to the start of other problems, such as that which **al- Samaw'al** indicates regarding the 'speed' of the differences. Without doubt the mathematicians had not yet invented the conceptual methods for posing these problems, but it did not make them less tempted to respond to certain of them, in an experimental way.[45]

The mathematicians not only pursued their research on these methods; they also applied them to disciplines other than astronomy. Thus Kamāl al-Dīn al-Fārisī had recourse to one of them—called *qaws al-khilāf,* the difference arc—to establish the table of refractions. Al-Fārisī proceeds thus: he divides the interval [0°, 90°] into two subintervals, where he approaches the function $f(i)=d/i$ (d the angle of deviation and i the angle of incidence) by an affine function on the interval [40°, 90°], and by a polynomial function of the second degree on the interval [0°, 40°]. He then connects the two interpolations.

But this method, called the 'difference arc' and applied by Kāmal al-Dīn al-Fārisī at the start of the fourteenth century, relates back to the mathematician al-Khāzin, from the tenth century, and will be taken up again then in the fifteenth century by al-Kāshī, in his *al-Zīj al-Khāqānī*. This illustrates well that, for this subject, it is a matter of stages of the same tradition. We shall concentrate for a short time on al-Kāshī.

Al-Kāshī wants to calculate the longitudes of the planets. He starts from a day with date 0, with a longitude λ_0. He then considers intervals of p days, assumes $\lambda_{-1}, \lambda_0, \lambda_p$ to be known and seeks to calculate the longitudes $\lambda_1, \lambda_2, \ldots, \lambda_{p-1}$ at the dates 1, 2,…, p^{-1}. We put

$\Delta_{-1}=\lambda_0-\lambda_{-1} \qquad \Delta_n=\lambda_{n+1}-\lambda_n$

and consider the arithmetic means of Δ on [0, *p*], i.e. $m_0(\Delta)=(\lambda_p-\lambda_0)/p$. If for the calculation of $\lambda_1, \lambda_2, \ldots, \lambda_{p-1}$ we take the mean increase $m_0(\Delta)$, we have a linear interpolation

$$\lambda_k=\lambda_0+km_0(\Delta)$$

But $m_0(\Delta)$ is very different from Δ_{-1}; one envisages then an interpolation of the second order. Al-Kāshī defines a number *e,* correction of the mean. One puts

$$q=\frac{p+1}{2} \qquad \text{and} \qquad e=\frac{m_0(\Delta)-\Delta_{-1}}{q}$$

If one considers the difference of order 2 as constant, this gives

$$\Delta_n^2=\Delta_{n+1}-\Delta_n=e$$

$$\Delta_m=\Delta_{-1}+(m+1)e$$

and

$$\sum_{m=0}^{k-1}\Delta_m=\lambda_k-\lambda_0=k\,\Delta_{-1}+\frac{k(k+1)}{2}\,e$$

One verifies that for $k=p$ one finds λ_p.

This step corresponds to increasing longitudes. If the longitudes are decreasing, one considers the absolute values of the differences, and the corrections are subtractive.

Such are the principal known methods of interpolation, and the principal problems posed. Alī show not only the importance of this area in the numerical analysis of this time, but also the distance covered by the mathematicians in the domain of calculation of finite differences.

INDETERMINATE ANALYSIS

The first research into indeterminate analysis—or, as it is called today, Diophantine analysis—in Arabic was conducted very probably in the middle of the ninth century, i.e. after al-Khwārizmī and before Abū Kāmil. In the book of the former, the indeterminate analysis did not figure 'in person', i.e. as a separate subject: even though al-Khwārizmī in the last part of his writing dedicated to the questions of inheritance and of partition started on some indeterminate problems, nothing indicates that he was interested in

Diophantine equations themselves. But the place that this analysis occupies later in the book by Abū Kāmil, written about 880, the level of the study of Abū Kāmil, as we shall see, the evocation, at last, by the latter, of other mathematicians who worked in this area, and that of their own terminology, does not leave any lingering doubt: Abū Kāmil was not the first, nor the only, successor of al-Khwārizmī to be actively occupied with the indeterminate equations. But the loss of writings constrains us therefore to start from the *Algebra* of Abū Kāmil, in order to follow first indeterminate rational analysis, in order then to show how it has become an area of algebra, before coming back to the description of the fact recently recognized as such: the constitution of the indeterminate integer analysis in a way counter the algebraists, as an integral part of the theory of numbers.

Rational Diophantine analysis

Abū Kāmil's project is clear; he writes: 'we are now explaining numerous indeterminate problems that arithmeticians call *sayyāla* ("fluids"), i.e. that one obtains for them numerous solutions with the aid of a convincing syllogism and a clear method. Certain of these problems circulate according to the type—*bi-al-abwāb*[46]—between the arithmeticians, without them having established the cause from which they proceed, and certain of these problems have been resolved by me with the help of a valid principle and an easy process'.[47] Abū Kāmil continues: 'we are also explaining with the aid of algebra and of syllogism, so that one who reads and examines it will truly understand it and not simply repeat it and imitate their author'.[48] This text is fundamental, historically and logically. It testifies to the existence of research into Diophantine analysis in the course of the half century that separates Abū Kāmil from al-Khwārizmī. The mathematicians who undertook this research dedicated the word *sayyāla* to the designation of Diophantine equations, in this way separating them with a term from the group of algebraic equations. Always according to the text by Abū Kāmil, we know that these mathematicians contented themselves with giving expositions of certain types of these equations and the algorithms to solve them, but without being preoccupied either with their reasons or with methods of establishing them. But who are these mathematicians? To this question, we can again not reply, because of the loss of the writings of numerous then active algebraists, such as ʿAlī, Abū Ḥanīfa al-Dīnawarī, Abū al- ʿAbbās etc.

Abū Kāmil therefore intends in his *Algebra* no longer to limit himself to a fragmented exposition, but to give a more systematic one, where, besides the problems and the algorithms of the solution, the methods will be given. Abū Kāmil, it is true, deals in the last part of his *Algebra* with thirty-eight Diophantine problems of the second degree, with four systems of indeterminate linear equations, with other systems of determined linear equations, with a group of problems which reduce to arithmetic progressions, and with a study of the latter.[49] This group answers to the double target fixed by Abū Kāmil: to solve indeterminate problems, and to solve by algebra problems dealt with at that time by the arithmeticians. We note that it is in *Algebra* by Abū Kāmil that one encounters for the first time in history—to my knowledge—an explicit distinction between determined problems and indeterminate problems. Now the examination of these thirty-eight Diophantine problems not only reflects this distinction; it also shows that these problems do not follow one another by chance, but according to an order subtly indicated by Abū

Kāmil. The first twenty-five are thus all taken from one and the same group, for which Abū Kāmil gives a sufficient and necessary condition to determine the positive rational solutions. Let us take just two examples. The first problem in this group[50] is rewritten

$$x^2+5=y^2$$

Abū Kāmil proposes to give two solutions among, according to his own declarations, the infinity of rational solutions. He suggests then

$$y=x+u \qquad \text{with } u^2<5$$

and takes successively $u=1$, $u=2$.

Another example from the same group is problem 19,[51] which is rewritten

$$8x-x^2+109=y^2$$

Abū Kāmil considers then the general form

$$ax-x^2+b=y^2 \tag{1}$$

and he writes:

> if you reach problems analogous to this problem, multiply half of the number of roots by itself and add this product to the dirhams (i.e. to the units); if the sum divides into two parts of which each has a square root, then the problem is rational and has countless solutions; but if the sum does not divide into two parts of which each has a square root, then the problem is irrational and without solutions.[52]

Particularly important in the history of Diophantine analysis, this text gives the condition sufficient to determine the positive rational solutions of the previous equation. This is rewritten

$$y^2+\left(\frac{a}{2}-x\right)^2=b+\left(\frac{a}{2}\right)^2$$

Putting $x=(a-t)/2$, we have

$$y^2+\left(\frac{t}{2}\right)^2=b+\left(\frac{a}{2}\right)^2 \tag{2}$$

and the problem is thus brought back to dividing a number into two other squares —problem 12 of the same group, and already solved by Abū Kāmil. Let us assume in effect that

$$b+\left(\frac{a}{2}\right)^2=u^2+v^2$$

with u and v rational. Abū Kāmil suggests

$$y=u+\tau$$
$$t=2(k\tau-v)$$

He substitutes in (2) and finds the values of y, t and then x. Thus he knows that, if one of the variables can be expressed as a rational function of the other, or, in other words, if one can have a rational parameterization, one has *all* the solutions; however, if the sum leads to an expression of which the radical is unavoidable there is no solution. In other words, unknown to Abū Kāmil, a curve of the second degree of genus zero possesses no rational point, or is birationally equivalent to a straight line.

The second group is made up of thirteen problems—26 to 38—which it is impossible to parameterize rationally; or, this time again in a language unknown to Abū Kāmil, they define all the curves of genus zero. Thus for example problem 31[53] is rewritten

$$x^2+x=y^2$$
$$x^2+1=z^2$$

which defines a quartic skew, curve of genus 1 in the affine space A^3.

The third group of indeterminate problems is made up of systems of linear equations, such as for example 39[54] which is rewritten

$$x+ay+az+at=u$$
$$bx+y+bz+bt=u$$
$$cx+cy+z+ct=u$$
$$dx+dy+dz+t=u$$

This interest in indeterminate analysis, which came about by the contribution of Abū Kāmil, led to another event: the translation of the *Arithmetica* of Diophantus. Thus, in the course of the same decade when Abū Kāmil wrote his *Algebra* in the Egyptian capital, Qusṭā ibn Lūqā translated seven Books of *Arithmetica* by Diophantus in Baghdad. The event was crucial, both for the development of indeterminate analysis and for the techniques of algebraic calculation. We have shown[55] that the Arabic version of *Arithmetica* is made up of three books, in common with the Greek text which reached us, and of four other books lost from the Greek text and for which the translation was done in the terminology invented by al-Khwārizmī. The translator not only gave an underlying algebraic interpretation to the *Arithmetica,* but he even gave to the book by Diophantus the title *Ṣinā' at al-Jabr—The Art of Algebra*. Now the Arabic version of *Arithmetica* has been studied and commented upon. We know at this time of the existence of four commentaries, of which three still remain untraceable. From the ancient bibliographers, we know that Qusṭā ibn Lūqā himself commented on three books of *Arithmetica*[56] and that Abū al-Wafā' al-Būzjānī wanted to prove the propositions, probably the algorithms, of Diophantus.[57] In his *al-Fakhrī,* al-Karajī[58] has commented on four books of *Arithmetica;* his successor al-Samaw'al has also commented on the book by Diophantus. Only the commentary by al-Karajī has reached us out of these four, which were not, we believe, the only commentaries on Diophantus. But, besides these commentaries, the algebraists have dealt with indeterminate analysis in their various writings, and this, with al-Karajī, will change its status.

Al-Karajī has himself dealt with Diophantine analysis in three writings, of which two have reached us. He studies indeterminate analysis in *al-Fakhrī,* before commenting Diophantus's book. He comes back onto this subject in *al-Badī'* and recalls in the introduction to this book his first work in *al-Fakhrī*. The third treatise has been put together with these two, but remains untraceable. It concerns, as he wrote in *al-Fakhrī,* a book 'on *al-Istiqrā'* [indeterminate analysis]' that he put together in the Persian province of Rayy, and he wanted an exhaustive book on this theme.[59]

In order to understand the contribution of al-Karajī to indeterminate analysis, it is necessary to bear in mind his renewal of algebra, which we have stressed before. Indeterminate analysis was in fact developed by al-Karajī as one of the subjects of algebra, and also as one of the means of extending algebraic calculation. Al-Karajī writes that Diophantine analysis 'is the pivot of the majority of calculations, and it is indispensable for all the subjects'.[60] Thus, after studying the polynomials which have square roots, and the way of extracting these roots, one goes on to algebraic expressions which only potentially have square roots. This is the main object of rational Diophantine analysis according to al-Karajī. It is in this sense that Diophantine analysis constitutes a subject of algebra. The method, or rather the methods, are those required to bring the problem back to an equality between two terms whose powers allow us to obtain rational solutions. Henceforth Diophantine analysis is given a proper name, *al-istiqrā'*,[61] a term which, itself also, emphasizes the duality because it designates a subject, and a method or a group of methods. In *al-Fakhrī,* al-Karajī defines this term thus: '*al-istiqrā'* in the calculation is that it brings you to an expression of one type or of two types or of three successive types (i.e. algebraic powers), which is not a square *in verbis* but is as regards the sense, and you can know its square root.'[62] In *al-Badī'*, al-Karajī takes up again the same definition and adds: 'I say that *al-istiqrā'* is the pursuit without respite of expressions, until you find what you are looking for.'[63]

A simple reading of the explanations of al-Karajī, as well as chapters dedicated to indeterminate analysis in two books, shows a certain split from his predecessors; the style of al-Karajī is different, not only from that of Diophantus, but also from that of Abū Kāmil. With regard to the difference from Diophantus, al-Karajī does not give himself ordered lists of problems and their solutions, but he organizes his exposition in *al-Badī'* around the number of terms from which the algebraic expression is composed and the difference between their powers. He considers for example in successive paragraphs:

$ax^{2n} \pm bx^{2n-1} = y^2 \quad ax^{2n} + bx^{2n-2} = y^2 \quad ax^2 + bx + c = y^2$

This principle of organization will be borrowed moreover by his successors. It is therefore clear that al-Karajī had as a target to give a systematic exposition. However, it takes further the task pioneered by Abū Kāmil, which consists in extricating as much as possible the methods for each class of problems. In *al-Fakhrī,* al-Karajī does not want to elaborate on the exposition of Diophantine analysis in the sense described, since, as we have remarked, he has already devoted a book to it, and he will come back to it in *al-Badīʿ* next. In *al-Fakhrī* he recalls only the principles of this analysis, indicating that it bears notably on the equation

$$ax^2 + bx + c = y^2 \qquad a, b, c \in Z \tag{1}$$

where the trinomial in x is not a square, in order finally to pass to different classes of problems of which the majority are indeterminate. These different classes are presented as classes of problems ordered from the easiest to the most difficult, in order to 'satisfy those who want to practise (*al-murtāḍ*).[64] It is in fact classes of exercises designed to familiarize the reader with the processes 'which lead the problem according to the expression of the asker to one of the six canonic forms to determine the unknowns from the knowns, which is the calculation in substance'.[65] In these five classes of problems, al-Karajī does not pretend any originality, and borrows the majority of the problems from books II, III and IV of *Arithmetica* by Diophantus, some problems from book I—as we have shown in detail elsewhere[66]—and more than half of the problems studied by Abū Kāmil. One also encounters other problems which do not feature among the work of these two authors, perhaps suggested by al-Karajī himself.

It is in *al-Badīʿ* that al-Karajī, in his own words, aims at a more informed and keen public than those to whom *al-Fakhrī* was aimed, that he lays out systematically the subject of Diophantine analysis. Thus, after having discussed the types mentioned previously, al-Karajī comes back to equation (1). He considers then the case where a (respectively c) is a square, and proposes the change of variable $y=\sqrt{a}x \pm u$ (respectively $y=\sqrt{c} \pm ux$). Note that he starts by giving the general formulation before proceeding to examples. He then produces the form

$ax^{2n} + bx^{2n-1} + c = y^2$

and proposes to reduce it to (1).

Al-Karajī next deals with the expressions where the powers succeed one another, such as

$ax^2 - c = y^2$

with a and c not squares but c/a square. He proposes the change of variable

$$y = ux - \sqrt{\left(\frac{c}{a}\right)}$$

Here also, he recalls that one can reduce by division the form

$$ax^{2n} - cx^{2n-2} = y^2$$

to the previous form.

Al-Karajī next studies equations of the form

$$ax^2 + c = y^2$$

and gives two examples, the first with $a=3$, $c=13$, and the second with $a=2$, $c=2$; and he remarks that, in these two examples, one has $a+c=k^2$. He proposes meanwhile the respective parameterization $y=u$ and $y=ux$, and obtains

$$x^2 = \frac{u^2 - c}{a} \qquad \text{and} \qquad x^2 = \frac{c}{u^2 - a}$$

which barely advances the solution of the problem. Commenting on this fact, Anbouba, in the French introduction to his critical edition of *al-Badīʿ*, writes fairly: 'But it is evident that al-Karajī ignores book VI by Diophantus which was forming the solution to the question (1) in the case $a+c$ equals a square (lemmas 1 and 2 of *Arithmetica* related to VI:12 and 13); (2) in the case where a particular root is known (lemma related to VI:15). We are almost convinced that al-Karajī was ignoring books V and VI of *Arithmetica* and the end of book IV.'[67]

Al-Karajī studied plenty of other problems, notably the double equality. We indicate simply the problem

$$x^2 + a = y^2$$
$$x^2 - b = z^2$$

which defines a curve of genus 1 in the affine space A^3.

The successors of al-Karajī did not only comment on his work, but were tempted to advance along the path carved out by him: to extend again the *al-istiqrā'* to certain cubic equations, and to extricate the methods. In this way in his *al-Bāhir* al-Samaw'al comments on *al-Badīʿ*, and includes in his definition of *al-istiqrā'* some equations of the form

$$y^3=ax+b$$

Al-Samaw'al affirms then that given that one of the terms on the righthand side is in a decimal position of the form $3k$, i.e. it can have a cubic root, the equation will always have solutions. Note that al-Samaw'al considers the case where $a=6$, $b=10$; now, for this value of a, whatever the value of b, the equation has a solution, since one has $y^3\equiv y$ (mod 6); but if $a=7$, then the equation $y^3=7x+2$ has no solutions, although it verifies the condition given by al-Samaw'al.

He next considers the equation

$$y^3=ax^2+bx$$

i.e. the case where none of the terms on the right-hand side are in a decimal position of the form $3k$. Al-Samaw'al then suggests finding a cubic number m^3 such that one of the two following conditions will be verified:

$$am^3+\left(\frac{b}{2}\right)^2=z^2 \quad \text{or} \quad bm^3+\left(\frac{a}{2}\right)^2=z^2$$

This barely advances the solution of the problem, since one is brought back to another problem that is no simpler.

It is not necessary to pursue here the works of the successors of al-Karajī on rational Diophantine analysis, but we note that henceforth it will form part of every treatise on algebra of any importance. Thus, in the first half of the twelfth century, al-Zanjānī borrowed most of the problems of al-Karajī and the first four books of the Arabic version of Diophantus; Ibn al-Khawwām set out certain Diophantine equations amongst which was Fermat's equation for $n=3$ $(x^3+y^3=z^3)$, just as Kāmal al-Dīn al-Fārisī did in his great commentary on the algebra of the latter. This interest and these works on indeterminate analysis carried on until the seventeenth century with al-Yazdī and, contrary to what the historians say about this subject, did not die out with al-Karajī.

Integer Diophantine analysis

The translation of the *Arithmetica* of Diophantus was not only essential to the development of rational Diophantine analysis as an area of algebra, but also contributes to the development of integer Diophantine analysis as a subject, not of algebra, but of the theory of numbers. In the tenth century, one witnessed for the first time the establishment of this subject, thanks to algebra without doubt but also opposed to it. The study of Diophantine problems was embarked upon by demanding on the one hand the obtaining of integer solutions and on the other proceeding by demonstrations of the Euclidean type as in his arithmetical books of the *Elements*. It is this combination, explicit for the first time in history—of the numerical domain restricted to positive integers interpreted as segments of straight lines, algebraic techniques and the requirement to prove in the pure Euclidean style—which has allowed the beginnings of this new Diophantine analysis. Rather than methods, the translation of the *Arithmetica* of Diophantus supplied these mathematicians, one understands, with certain problems of the theory of numbers, which they did not hesitate to systematize and examine for themselves, contrary to what one can see in the work of Diophantus. These are for example the problems of the representation of a number as a sum of squares, congruent numbers etc. In brief, one encounters here the start of the new Diophantine analysis in the sense that one will find it again later in the work of Bachet de Méziriac and Fermat.[68] It is somewhat surprising that such an event has escaped the attention of historians, even of those who have knowledge of certain works of these mathematicians.[69] Other historians of mathematics, faced with this deficiency, can only condemn the theory of numbers in Arabic mathematics as seriously lacking. Perhaps the main reason for the ignorance about this subject is to be found in the absence of a historical perspective, which would have showed that this research into integer Diophantine analysis was not the work of one lone mathematician but of a whole tradition which included, besides al-Khujandī and al-Khāzin, al-Sijzī, Abū al-Jūd ibn al-Layth, Ibn al-Haytham, as well as later mathematicians such as for example al-Samaw'al, Kamāl al-Dīn ibn Yūnus, al-Khilāṭī, al-Yazdī, etc.

The authors of the tenth century themselves emphasized this novelty. Thus one of them, after having given the principle for the generation of numerical right-angled triangles, writes: This is the basis of the knowledge of hypotenuses of primitive right-angled triangles. I have not found that this has been mentioned in any of the books of the ancients, and none of those who have written arithmetical books among the modern writers has stated it, and I know that this has been revealed to none of my predecessors'.[70] In this anonymous writing, as in others from the hand of al-Khāzin—one of the founders of this tradition—the mathematicians have introduced the fundamental concepts of this new analysis: that of the primitive right-angled triangle (*aṣl al-ajnās*), of the generator and, above all, of the representation of a solution with respect to a certain modulus. It is true that the new domain is organized around the study of numerical right-angled triangles and congruent numbers, as well as a variety of problems on the theory of numbers linked to these two themes.

The author of the anonymous text previously cited, after introducing the basic concepts for the study of Pythagorean triangles, ponders over the integers that can be the hypotenuses of these triangles; i.e. the integers that one can represent as the sum of two squares. He states in particular that every element of the series of primitive Pythagorean triplets is such that the hypotenuse is of one or other form: 5 (mod 12) or 1 mod (12). He notes meanwhile—as al-Khāzin after him—that certain numbers of this series −49 and 77 for example—are not the hypotenuses of such triangles. This same author also knew that certain numbers of the form 1 (mod 4) cannot be hypotenuses of primitive right-angled triangles.

Al-Khāzin gives next the analysis of the proposition proved only by synthesis in the *Elements,* lemma 1 to X−29.

Let (x, y, z) be a triplet of integers such that $(x, y)=1$, x even. The following conditions are equivalents.

1 $x^2+y^2=z^2$
2 There exists a pair of integers $p>q>0$; $(p, q)=1$ and p and q are of different parity, such that

$$x=2pq \qquad y=p^2-q^2 \qquad z=p^2+q^2$$

Al-Khāzin then solves the equation:[71]

$$x^2=x_1^2+x_2^2+\cdots+x_n^2$$

His reasoning is general, even if he stops at the case $n=3$. He considers next two equations of the fourth degree:

$$x^2+y^2=z^4 \qquad \text{and} \qquad x^4+y^2=z^2$$

Without lingering any more on these studies on the numerical triangles investigated by al-Khāzin, and later by Abū al-Jūd ibn al-Layth, we come to the problem of congruent numbers, i.e. to solutions of the system

$$\begin{aligned} x^2+a&=y_1^2 \\ x^2-a&=y_2^2 \end{aligned} \qquad (1)$$

The author of the anonymous text gave the identities

$$(u^2+v^2)^2\pm 4uv(u^2-v^2)=(u^2-v^2\pm 2uv)^2 \qquad (2)$$

which allow the solution of (1) if $a=4uv(u^2-v^2)$. These identities can be deduced directly from the following:

$$z^2\pm 2xy=(x\pm y)^2$$

By substituting

$$x=u^2-v^2 \qquad y=2uv \qquad z=u^2+v^2$$

one obtains (2).

Al-Khāzin proves then the following theorem.

Let a be a given natural integer. The following conditions are equivalent:

1 System (1) has a solution.
2 There exists a couple of integers (m, n) such that

$$m^2+n^2=x^2$$

$$2mn=a$$

In these conditions, a is of the form $4uv(u^2-v^2)$.

It is in this tradition that the study of the representation of an integer as the sum of squares has been undertaken. Thus, al-Khāzin devotes several propositions in his work to this study. In the course of this important research, he shows a direct knowledge of 111–19 of *Arithmetica* by Diophantus—and therefore of the Arabic version of this book—and of the identity already encountered by the ancient mathematicians:

$$(p^2+q^2)(r^2+s^2)=(pr\pm qs)^2+(ps\mp qr)^2$$

Al-Khāzin also looks for integer solutions of the system of Diophantine equations, such as 'to find four different numbers, such that their sum is a square, and that every sum of two among them is a square',[72] i.e.

$$x_1+x_2+x_3+x_4=y^2$$

$$x_i+x_j=z_{ij}^2 \qquad (i<j)\left[\binom{4}{2}\text{equations}\right]$$

It is also these mathematicians who were the first to pose the question of impossible problems, such as the first case of the theorem by Fermat. It had been known for a long time that al-Khujandī had tried to prove that 'the sum of two cubic numbers is not a cube'. According to al-Khāzin,[73] the proof by al-Khujandī is defective. A certain Abū Jaʿfar also attempted to prove the following proposition: 'it is impossible that the sum of two cubic numbers is a cubic number, although it is possible that the sum of two square numbers will be a square number; and it is impossible that a cubic number will divide into two cubic numbers, although it is possible that a square number will divide into two square numbers'.[74]

The proof by Abū Jaʿfar is also defective. Even though it was necessary to wait for Euler in order for this proof to be established, the problem did not cease, despite everything, to preoccupy the Arab mathematicians who, later, stated the impossibility of the case $x^4+y^4=z^4$.

Research into integer Diophantine analysis and notably numerical right-angled triangles did not stop with its initiators in the first half of the tenth century. Quite the opposite—its successors took it up again, and in the same spirit, in the second half of the same century and at the start of the following century, as prove the examples of Abū al-Jūd ibn al-Layth, al-Sijzī and Ibn al-Haytham. Others, later, such as Kamāl al-Dīn ibn Yūnus, pursued this research in one way or another. We start by considering briefly the writings of Abū al-Jūd and al-Sijzī.

In a treatise on numerical right-angled triangles, Abū al-Jūd ibn al-Layth takes up the problem of their formation and the conditions necessary for the formation of primitive triangles, and above all establishes tables to draw, starting from pairs of integers $(p, p+k)$, with $k=1, 2, 3, \ldots$, the sides of triangles obtained, their areas and the ratio of these areas to the perimeters. He also comes back at the end to the problem of congruent numbers.

His junior al-Sijzī also busied himself with triangles, and notably with the solution of the equation

$$v^2 = x_1^2 + \cdots + x_n^2 \qquad (*)$$

His procedure consists of researching the smallest integer t such that

$$2vt=z^2$$

from which he gets

$$(v+t)^2 = x_1^2 + \cdots + x_n^2 + t^2 + z^2$$

and thus finds a number which is the sum of $(n+2)$ squares. He shows that if one knows how to solve the cases $n=2$ and $n=3$, one can solve the general case.

In fact, with the help of a complete finite induction, slightly archaic, al-Sijzī demonstrates the following proposition:

(p_n): for all n, there exists a square which is the sum of n squares.

Thus he first proves the case p_2, i.e.

$$x^2+y^2=z^2$$

by analysis and synthesis. His analysis reverts in fact to showing geometrically that

$$y^2=(z-x)(z+x)$$

In the synthesis, he takes the even term, y^2 say,

$$y^2=2^k b(2a)$$

Then $z+x$ is even and one has

$$z-x=2^k b \quad \text{and} \quad z+x=2a$$

One finds

$$z=a+2^{k-1}b \quad \text{and} \quad x=a-2^{k-1}b$$

and thus one obtains a solution for each k such that

$$k>0 \quad 2^{k-1}b<a$$

i.e.

$$y^2>2^{2k}b^2 \quad y>2^k b \quad y^2=2^{k+1}ab$$

In particular, if b=1, one has

$$y^2=2^{k+1}a \quad 2\leq 2^k<y$$

whence one solution if y is divisible by 2 and y>2, three solutions if y is divisible by 4 and y>8, and more generally $2h-1$ solutions if y is divisible by 2^h and $y>2^{2h+1}$.

Thus, in this case, al-Sijzī proves that for n=2 there exists a square which is the sum of two squares in several ways.

In the case p_3, i.e.

$$x^2+y^2+z^2=t^2$$

al-Sijzī introduces a condition which renders the construction less general, i.e. $t=x+y$. He then shows that, if one has p_n, then one has p_{n+2}, whence a recurrence for n even and a recurrence for n odd.

Al-Sijzī gives a table, up to n=9, that is reproduced here as Table 12.2. It can be seen that al-Sijzī has constructed the table by means of the rule of recurrence.

As can be seen, the works of Abū al-Jūd ibn al-Layth and al-Sijzī on integer Diophantine analysis are very much in the tradition of al-Khāzin; they borrow from him the principal problems, strengthening to some extent the geometrical methods of proof, which establishes the gap between algebra and rational Diophantine analysis. It remains that in the tradition of al-Khāzin and his predecessors, besides the deliberate use of the Euclidean language of segments to provide proofs in this area, they bring to the fore arithmetical arguments such as those designed to show that every element in the series of primitive Pythagorean triplets is such that the hypotenuse is of one or the other form 5 (mod 12) or 1 (mod 12). It is precisely in this direction that Diophantine analysis seems to

Table 12.2

Number of square roots						*Squares*					*Total*
[n=]	2	64	36								100
	10										
	3	36	81	4							121
	11										
	4	36	64	400	400						900
	30										
	5	4	4	1	36	36					81
	9										
	6	900	64	36	400	400	225				2025
	45										
	7	4	4	1	36	36	36	4			121
	11										
	8	900	64	36	400	400	225	900	100		3025
	55										
	9	4	4	1	36	36	36	4	484	484	1089
	33										

have evolved already in Arabic mathematics, before taking it up completely from Fermat. Instead of geometrical language, one wanted to proceed with purely arithmetical methods. We still do not know exactly when this important change of direction occurred, but we encounter it in the works of later mathematicians. Thus al-Yazdī devotes a short paper to the solution of the same Diophantine equation (*) with the help of purely arithmetical methods; he studies the different cases as a function of the parity of x_i and he systematically uses a calculation equivalent to the congruences modulo 4 and modulo 8.[75] Among the multiple lemmas that he proves, we will cite two in order to illustrate the procedure and the style.

Let n be odd but $n \not\equiv 1$ (mod 8); then $x_1^2 + \cdots + x_n^2$ cannot be a square if $x_1, \ldots, x_n$ are odd numbers.

Let n be odd and $n \equiv 1$ (mod 8); if $x_1, \ldots, x_{n-1}$ are given odd numbers, there exists x_n odd such that $x_1^2, \cdots, x_n^2$ is a square.

It is with the help of lemmas of this type that he establishes equation (*).

Several results from the works of mathematicians have been transmitted and are found in the *Liber quadratorum* and, occasionally, in the *Liber abaci* by Fibonacci; but the subject will be renewed thanks to the invention of the method of infinite descent by Fermat.

THE CLASSICAL THEORY OF NUMBERS

The contribution of the mathematicians of this time to the theory of numbers was not limited to integer Diophantine analysis. Two other movements of research, starting from different points, led to the extension and the renewing of the Hellenistic theory of numbers. The first movement had as its source, but also its model, the three arithmetical books of *Elements* by Euclid, while the second is situated in the neo-Pythagorean lineage, as it appears in the *Introduction to Arithmetic* by Nicomachus of Gerasa. It is in the Euclidean books that one finds a theory of parity and the multiplicative properties of integers: divisibility, prime numbers etc. For Euclid, an integer is represented by a segment of a straight line, a representation essential for the proving of propositions. Although the neoPythagoreans shared this concept of integers and considered principally the study of the same properties, or properties derived from them, by their methods and their aims they distinguish themselves from Euclid. While the latter proceeded through proofs, the former worked solely by means of induction. However, for Euclid arithmetic had no other goal outside itself, while for Nicomachus it had philosophical, and even psychological, ends. This difference in method was clearly recognized by the Arab mathematicians such as Ibn al-Haytham, who wrote: ‘The properties of numbers are shown in two ways; the first is induction, because if one follows the properties of the numbers one by one and if one distinguishes them, one finds all their properties by distinguishing them and considering them. This is shown in the work ***al-Arithmāṭīqī*** [*Introduction to Arithmetic* by Nicomachus]. The other way by which the properties of numbers are shown proceeds by proofs and deductions. Alī the properties of number gained by proofs are contained in these three books [of Euclid] or in what relates to them’.[76] For the mathematicians of the time, it was therefore a difference between the methods of demonstration, and not between the objects of arithmetic. One therefore understands that, despite a marked preference for the Euclidean method, the mathematicians, even those as important as Ibn al-Haytham, proceeded in certain cases by induction, according to the problem posed; it is thus that Ibn al-Haytham discusses the ‘Chinese theorem’ and the Wilson theorem. However, although mathematicians of first order, and certain philosophers such as Avicenna, neglect philosophical and psychological aims, ascribed by Nicomachus to arithmetic; other mathematicians of an inferior standard, philosophers, doctors, encyclopaedists etc. are interested in this arithmetic. The history of it is therefore founded in that of the culture of the well-read of Islamic society over centuries, and extends well beyond the scope of this book. We are deliberately limiting ourselves to the participation of arithmetic in the development of the theory of numbers as a discipline.

Research into the theory of numbers in the Euclidean and Pythagorean sense started early, before the end of the ninth century. It was at the same time as the translation by

Thābit ibn Qurra of the book by Nicomachus and the revision of the translation of the *Elements* of Euclid by the same man. It is Thābit ibn Qurra (d. 901) who started this research into the theory of numbers, by expounding the first theory of amicable numbers. This fact, known to historians since the last century thanks to the work of Woepcke,[77] only found its true meaning recently when we established the existence of a whole tradition, unveiled by Thābit ibn Qurra in the purest Euclidean style and added to some centuries later by al-Fārisī (d. 1319), thanks to the application of algebra to the study of the first elementary arithmetical functions; this tradition has been marked by plenty of names: al-Karābīsī, al- Anṭākī, al-Qubayṣī, Abū al-Wafā' al-Būzjānī, al-Baghdādī, al-Karajī, Ibn al-Haytham, Ibn Hūd, etc., to name but a few. One cannot of course pretend, in the few pages devoted to this theory, to give a detailed description. We will simply try to outline the movement that we have just mentioned.

Amicable numbers and the discovery of elementary arithmetical functions

At the end of book IX of *Elements,* Euclid gives a theory of perfect numbers and proves that the number $n=2^P(2^{P+1}-1)$ is perfect—i.e. equal to the sum of its proper divisors—if $2^{P+1}-1$ is a prime number. But Euclid did not try, any more than Nicomachus or any other Greek author, to elaborate on a similar theory for amicable numbers. Thābit ibn Qurra therefore decides to construct this theory. He sets out and proves, in pure Euclidean style, the most important theorem so far of these numbers, which today bears his name.

Note that $\sigma_0(n)$ is the sum of aliquot parts of the integer n, and $\sigma(n)=\sigma_0(n)+n$ is the sum of divisors of n; and recall that two integers a and b are called amicable if $\sigma_0(a)=b$ and $\sigma_0(b)=a$.

Theorem of Ibn Qurra

For $n>1$, put $p_n=3.2^n-1$, $q_n=9.2^{2n-1}-1$; if p_{n-1}, p_n and q_n are prime, then $a=2^n p_{n-1} p_n$ and $b=2^n q_n$ are amicable.

The proof that Ibn Qurra outlines relies on a proposition equivalent to IX– 14 of *Elements,*[78] to exploit the properties of the geometric series of ratio 2.

Now, the history of the arithmetical theory of amicable numbers from Ibn Qurra until at least the end of the seventeenth century limits itself to an evocation of this theorem, to its transmission through later mathe-maticians and to the calculation of pairs of these numbers. Among the mathematicians of the Arabic language, from a very long list we recall al-Anṭākī (d. 987), al-Baghdādī, al-Karajī, Ibn Hūd, Ibn al-Bannā', al-Umawī.[79] These few names, to which we will add many others, show the widespread diffusion of the theorem of Ibn Qurra by both their chronological diversity and their geographical diversity; the theorem is found again in 1638 in the work of Descartes. But, for him as for his Arab predecessors, it seems to be obvious that Ibn Qurra's theorem is exhaustive.

As regards the calculation of pairs of amicable numbers, Ibn Qurra does not make the effort to calculate any other pair than (220, 284), not because of an inability to find others but because of the little interest that this Euclidean gives to such a calculation. Neither does al-Anṭākī, three-quarters of a century later, seem to have calculated others. It is notably by the algebraists that this calculation is undertaken. Thus, one finds amongst the work of al-Fārisī in the East, around Ibn al-Bannā' in the West, of al-Tanūkhī and of many other thirteenth-century mathematicians, the pair (17296, 18416), known as Fermat. Al-Yazdī later calculated the pair known as Descartes (9363584, 9437056).

Such a historical summary, the most complete that exists at present, remains nevertheless both abridged and blind: it ignores in effect the role played by the research into amicable numbers in the whole of the theory of numbers, just as it ignores the intervention of algebra into the latter. We shall not linger on the previously mentioned works, in order to present this intervention of algebra. The famous physicist and mathematician Kamāl al-Dīn al-Fārisī compiled a paper in which he set out deliberately to prove the theorem of Ibn Qurra in an algebraic way. This forced him to an understanding of the first arithmetical functions, and to a full preparation which led him to state for the first time the fundamental theorem of arithmetic. Al-Fārisī also developed the combinatorial methods necessary for this study, and thus a whole research into figurate numbers. In short, it is a matter this time of the elementary theory of numbers, such as is found again in the seventeenth century.

Throughout his paper, al-Fārisī accumulates the propositions necessary for characterization of the two first arithmetical functions: the sum of the divisors of an integer, and the number of these divisors. The paper starts with three propositions, of which the first says: 'Every composite number is necessarily decomposable into a finite number of prime factors, of which it is the product.' In the other propositions, he tries, though clumsily, to prove the uniqueness of the decomposition.

Unlike Ibn Qurra's text, the exposition of al-Fārisī does not open with a proposition equivalent to IX–14 of Euclid, much less with IX–14 itself; but the author expounds in turn the existence of a finite decomposition into prime factors and the uniqueness of this decomposition. Thanks to this theorem, and to the combinatorial methods, one can completely determine the aliquot parts of a number, i.e. in the terms of al-Fārisī in addition to the prime factors, 'every number composed of two of these factors, three of these factors and so on until every number composed of all these factors less one'.

Following these propositions, al-Fārisī examines the processes of factorization, and the calculation of aliquot parts as a function of the number of prime factors. The most important result is without doubt the identification between the combinations and the figurate numbers. Thus, all is henceforth in place for the study of arithmetical functions. A first group of propositions concerns $\sigma(n)$. Although al-Fārisī only in fact deals with $\sigma_0(n)$, one notes that he recognized σ as a multiplicative function. Amongst the propositions in this group, the following are notable.

1 If $n=p_1p_2$, with $(p_1, p_2)=1$, then

$$\sigma_0(n)=p_1\sigma_0(p_2)+p_2\sigma_0(p_1)+\sigma_0(p_1)\sigma_0(p_2)$$

which shows that he knows the expression

$$\sigma(n)=\sigma(p_1)\sigma(p_2)$$

2 If $n=p_1p_2$, with p_2 a prime number and $(p_1, p_2)=1$, then

$$\sigma_0(n)=p_2\sigma_0(p_1)+\sigma_0(p_1)+p_1$$

3 If $n=p^r$, with p a prime number, then

$$\sigma_0(n)=\sum_{k=0}^{r-1} p^k=\frac{p^r-1}{p-1}$$

These three propositions have until now been attributed to Descartes.

4 Finally he tries, but without succeeding, as one can easily understand, to establish an effective formula for the case where $n=p_1p_2$, with $(p_1, p_2)\neq 1$.

A second group includes several propositions bearing on the proposition $\tau(n)$: the number of divisors of n.

5 If $n=p_1p_2..., p_r$, with $p_1,..., pr$ different prime factors, then the number of aliquot parts of n, denoted $\tau_0(n)$, is equal to

$$1+\binom{r}{1}+\cdots+\binom{r}{r-1}$$

This proposition is attributed to Deidier (1739).

6 If

$$n=p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$$

then

$$\tau(n)=\prod_{i=1}^{r}(e_i+1)$$

and $\tau_0(n)=\tau(n)-1$; this proposition is attributed to John Kersey and to Montmort.

Al-Fārisī finally proves the theorem of Thābit ibn Qurra. It is necessary for him simply to show that

$$\sigma(2^n p_{n-1} p_n)=\sigma(2^n q_n)=2^n(p_{n-1}p_n+q_n)=9.2^{2n-1}(2^{n+1}-1)$$

This brief analysis of the paper by al-Fārisī shows the appearance of a new style, planted in old ground, that of the theory of numbers. Without leaving the ground of Euclid, the thirteenth-century mathematicians did not hesitate to turn back to the contributions of algebra, and notably combinatorial analysis. Now this tendency appears again, when mathematicians like al-Fārisī and Ibn **Ibn al-Bannā'** study figurate numbers, as we have seen previously.[80]

Perfect numbers

If, with the work on amicable numbers, mathematicians were also seeking to characterize this class of integers, in studying perfect numbers they were pursuing the same goal. We know through the mathematician al-Khāzin that in the tenth century the existence of odd perfect numbers was being questioned—a problem that is still not solved.[81] At the end of the same century and the beginning of the following, al-Baghdādī[82] obtained some results concerning these same numbers. Thus he gives the following.

> If $\sigma_0(2^n)=2^n-1$ is prime then $1+2+\ldots+(2^n-1)$ is a perfect number, a rule attributed to the seventeenth-century mathematician J.Broscius. A contemporary of al-Baghdādī, Ibn al-Haytham,[83] was the first to try to characterize this class of even perfect numbers, by trying to prove the following theorem.
>
> Let n be an even number. The following conditions are equivalent:
>
> 1 If $n=2^p(2^{p+1}-1)$, with $2^{p+1}-1$ prime, then $\sigma_0(n)=n$.
>
> 2 If $\sigma_0(n)=n$, then $n=2^p(2^{p+1}-1)$, with $2^{p+1}-1$ prime.

It is known that condition 1 is none other than IX–36 from the *Elements* by Euclid. Ibn al-Haytham therefore tries to prove as well that every even perfect number is of the Euclidean form, a theorem which will be established definitively by Euler. Note that Ibn al-Haytham does not try for perfect numbers, any more than Thābit ibn Qurra does for amicable numbers, to calculate numbers other than those known and transmitted by tradition. This calculator task will be one for mathematicians of an inferior class, closer to the tradition of Nicomachus of Gerasa, such as Ibn Fallus (d. 1240) and Ibn al-Malik al-Dimashqī,[84] among many others. Their writings teach us that the mathematicians at that time knew the first seven perfect numbers.

The characterization of prime numbers

One of the axes of research into the theory of numbers has therefore been the characterization of numbers: amicable, equivalent,[85] perfect. One must not be amazed in these conditions that mathematicians revert to prime numbers in order to proceed to a similar task. It is precisely what Ibn al-Haytham did in the course of solving the problem called 'of the Chinese remainder'.[86] He wants to solve the system of linear congruences

$$x \equiv 1 \pmod{i_i}$$
$$x \equiv 0 \pmod{p}$$

with p a prime number and $1 < i_i \leq p-1$.

In the course of this study, he gives a criterion for determining prime numbers, or the so-called Wilson theorem:

For $n>1$, the two following conditions are equivalent.

1 n is prime
2 $(n-1)! \equiv -1 \pmod{n}$

or, in the words of Ibn al-Haytham: 'this property is necessary for every prime number, that is to say that for every prime number—which is the number which is only a multiple of unity—if one multiplies the numbers which precede it by one another in accordance with the way we have already introduced, and if one is added to the product, then if the sum is divided by each of the numbers which precede the prime number, one remains, and if one divides by the prime number, nothing is left.'[87]

The study of this system of congruences is found in part in the work of the successors of Ibn al-Haytham in the twelfth century, al-Khilāṭī in Arabic and Fibonacci in Latin[88] for example.

To these areas of theory in Arabic mathematics, one could add a multitude of results which fit the pattern of the arithmetic of Nicomachus, developed by arithmeticians and algebraists, or simply for the needs of other practices such as magic squares or arithmetic games. One recalls as a matter of interest the sums of the powers of natural integers, polygonal numbers, the problems of linear congruences etc. There is thus a considerable number of results, which extend or prove what was already known, but which it is impossible to add here.[89]

NOTES

1 In effect, among the areas of Arabic mathematics that the historians have usually studied, one will search in vain for those which deal with combinatorial analysis, with integer Diophantine analysis, and with the classical theory of numbers. It is only in our study published in 1973 that the status of this activity was examined and that this first area was recognized as such. The same for integer Diophantine analysis: it has never been described as an activity independent from indeterminate analysis or from rational Diophantine analysis before our study published in 1979, It is the same also for the classical theory of numbers, and the role of algebra in its formulation has only been recognized in our study of 1983. These studies, grouped

together with others, are published in Rashed (1984).
2 Rashed (1973).
3 In his voluminous *The Codebreakers,* David Kahn writes: 'Cryptology was born among the Arabs. They were the first to discover and write down the methods of cryptoanalysis' (p. 93). This fact, known for a long time, has recently been emphasized because of the development of the theory of codes. The book by Ibn Waḥshiyya was translated into English by Joseph Hammer in 1806, under the title *Ancient Alphabets and Hieroglyphics Explained.* Cf. also Bosworth (1963).
4 Al-Samaw'al, pp. 104 et seq.
5 *Ibid.,* p. 232 of the Arabic text and pp. 77 *et seq*. of the introduction.
6 See our forthcoming study 'Métaphysique et combinatoire'.
7 Naṣīr al-Dīn al-Ṭūsī, 'Jawāmiʿ al-ḥisāb', pp. 141–6.
8 Rashed (1982, 1983).
9 See the preceding note.
10 Rashed (1982:210).
11 Al-Kāshī, pp. 73–4, where he gives the law for the composition of the arithmetical triangle.
12 *Al-isʿāf al-atamm*, MS Riyāḍa 182, Dār al-Kutub, Cairo; he gives the arithmetical triangle and explains the formation of it on pages 46–7. In the triangle, al-Dimashqī puts the names—thing, square etc.—in abbreviation.
13 *ʿUyūn al-Ḥisāb*, MS Hazinesi 1993, Süleymaniye, Istanbul; see the arithmetical triangle on folios 1 and 20^{r}–^{v}.
14 *Bughyat al-ṭullāb*, MS 496, col. Paul Sbath, fol. 137^{v}–138^{r}.
15 See our forthcoming study 'Métaphysique et combinatoire'.
16 Luckey (1948).
17 Rashed (1978a).
18 Cf. Sharaf al-Dīn al-Ṭūsī, vol. I, pp. lix–cxxxiv.
19 This book is for the time being only known through its Latin version. See on this subject the chapter by Allard in the present work.
20 Cf. al-Baghdādī, p. 76.
21 Banū Mūsā, *Rasāʾil al-Ṭūsī*, vol. 2, p. 25; cf. also the Latin translation in Clagett (1964: vol. I, p. 350) and the commentary by the editor, p. 367.
22 Al-Uqlīdisī (1st edition), pp. 218 and 313–14.
23 Al-Baghdādī, *al-Takmila,* pp. 76–80 and 84–94.
24 Cf. Kūshyār ibn Labbān.
25 See our work on the extraction of the square root and cubic root by Ibn al-Haytham in Rashed (1993c), Appendix.
26 Suter (1906). See also al-Nasawī, pp. 65 *et seq.* of the introduction to the Persian version, and pp. 8 *et seq.* of the photocopy of the published Arabic text.
27 Naṣīr al-Dīn al-Ṭūsī, 'Jawāmiʿ al-ḥisāb', pp. 141 *et seq*. and 266 *et seq*.
28 Ibn al-Khawwām, *Al-Fawāʾid al-Bahāʾiyya fī al-Qawāʿid al-Ḥisābiyya,* MS Or.5615, The British Library, 7^{V} and 8^{r}.
29 Cf. Kamāl al-Dīn al-Fārisī.
30 See note 8.

31 MS 7473, British Library. See particularly from 367^r to 374^r.
32 See note 8.
33 See the chapter on numeration and arithmetic.
34 Al-Uqlīdisī (1st edition), p. 145. See also the English translation by Saidan.
35 Rashed (1978a).
36 Al-Samaw'al, *al-Qiwāmī fī al-Ḥisāb al-Hindī,* fol. 111^v, in Rashed (1984:142).
37 *Ibid.,* p. 122.
38 Cf. al-Kāshī, pp.79 and 121, Luckey (1951:103). Cf. Rashed (1984:132 *et seq.*).
39 *Bughyat al-ṭullāb,* fol. 131^r et seq.
40 In the treatise by al-Yazdī, *'Uyūn al-ḥisāb,* one cannot help but notice a certain familiarity with decimal fractions, while he prefers to calculate with sexagesimal fractions and ordinary fractions; cf. for example fol. 9^V, 49^r–^{v}.
41 Al-Kāshī introduces a vertical line which separates the fractional part; this representation is found in the work of Westerners such as Rudolff, Apian, Cardano. The mathematician Mīzrahī (born in Constantinople in 1455) used the same sign before Rudolff. As regards the Byzantine manuscript, one notes particularly: The Turks carried out the multiplications and the divisions on the fractions according to a specific procedure of calculation. They had introduced their fractions while governing our land here.' The example given by the mathematician leaves no doubt about the fact that it concerns decimal fractions. See Herbert Hunger and Kurt Vogel (1963), p. 32 (problem 36).
42 Neugebauer (1957), p. 28.
43 Hamadanizadeh (1978).
44 Rashed (1991a).
45 *Ibid.*
46 The term *bāb* is used with this double meaning at the time, as is evident in the algebra of al-Khwārizmī for example. It thus signifies in one sense 'type', and is synonymous with *ḍarb*; al-Khwārizmī writes in this meaning: 'We have found that all that is done by the calculation of algebra and of *al-muqābala* must lead you to one of the six types—*Abwāb*—that I have described in my book' (Al-Khwārizmī, *Kitāb al-jabr,* p. 27). Here it is therefore the meaning 'type'. This term could also mean 'algorithm'. Thus, after having stated the type 'two squares plus roots equal a number', he gives the example $x^2+10x=39$, and he writes: 'The rule—*fabābuhu*—is to share into two halves the number of roots, five in this problem, that you multiply by itself—which makes twenty-five—that you add to thirty-nine, which makes sixty-four; you take its root, which is eight, from which you subtract half the roots, which is five; three remains, the root of the square.' Finally, a third meaning, which is the current meaning and also used at that time, is 'chapter'. These uses are also found in the algebra of Abū Kāmil.
47 Abū Kāmil, *Kitāb fī al-jabr wa al-muqābala,* 79^r.
48 *Ibid.*
49 This part occupies folios 79^r–110^v.
50 *Ibid.,* 79^r–79^v.

51 *Ibid.,* 87^{r}–87^{v}.

52 *Ibid.,* 87^{r}.

53 *Ibid.,* 92^{v}.

54 *Ibid.,* 95^{r}–95^{v}.

55 Diophantus, *Les Arithmétiques,* vol. III. See also Rashed (1974a): the introduction to the edition princeps of Diophantus, *Ṣinā'at al-Jabr,* p. 13 *et seq.* of the introduction.

56 Diophantus, *Les Arithmétiques,* pp. 10–11.

57 *Ibid.*

58 Franz Woepcke (1853); cf. also the translation of the problems from book IV of Diophantus taken up by al-Karajī in the complementary notes to *Arithmetica* corresponding to this book.

59 *Ibid.,* p. 74. It is necessary to correct the reading by Woepcke, and to read *bi-al-Rayy* and not *bi-al-tattary*.

60 Cf. al-Karajī, Arabic text p. 8.

61 This term is derived from the verb *istaqrā'* which means 'to consider or examine successively the different cases', before taking the technical meaning of indeterminate analysis.

62 Woepcke (1853:72).

63 Al-Karajī, p. 62.

64 *Al-Fakhrī,* MS Köprülü 950, fol. 54^{r}.

65 *Ibid.*

66 Cf. Diophantus, *Ṣinā'at al-Jabr,* introduction, pp. 14–19.

67 This remark was first made by A.Anbouba, editor of *al-Badī'*.

68 Rashed (1979).

69 Rashed (1983).

70 Rashed (1979:201–2).

71 See note 68.

72 Rashed (1979).

73 *Ibid.,* p. 220.

74 *Ibid.,* p. 222.

75 This text, as well as those of Abū al-Jūd ibn al-Layth and of al-Sijzī, are the object of separate research, to be published. See also Rashed (1974).

76 Ibn al-Haytham, *Sharḥ muṣādarāt Uqlīdis,* MS Feyzullah, Istanbul 1359, 213^{v}.

77 Woepcke (1852). In this text, Woepcke summarizes the pamphlet by Ibn Qurra.

78 This proposition is written: 'If a number is the smallest that can be measured by prime numbers, it will not be measured by any other prime number, if it is not by those which first measured it'; in other words, the lowest common multiple of prime numbers does not have any prime divisors other than these numbers.

79 See Rashed (1982:209–18; 1983; 1989).

80 *Ibid.*

81 Al-Khāzin writes: 'This question is posed to those who question themselves [about abundant, deficient and perfect numbers] whether or not a perfect number exists among the odd numbers or not.' See the Arabic text studied by Anbouba, p. 157.

82 Rashed (1984), p. 267.

83 Rashed (1989).

84 *Ibid.*

85 The numbers equivalent to *a* are the numbers defined by $\sigma_0^{-1}(a)$, i.e. the numbers for which the sum of the proper divisors of each is equal to *a*. Thus if *a*=57, one has $\sigma_0^{-1}(57) = \{159, 559, 703\}$.

86 Rashed (1984:238).

87 Rashed (1980; 1984:242).

88 *Ibid.*

89 It is a matter of reading the arithmetical works of arithmeticians such as al-Uqlīdisī, al-Baghdādī, al-Umawī etc.; of algebraists such as Abū Kāmil, al-Būzjānī, al-Karajī, al-Samaw'al; of philosophers such as al-Kindī, Ibn Sīnā, al-Juzjānī etc., among hundreds of others.

13
Infinitesimal determinations, quadrature of lunules and isoperimetric problems

ROSHDI RASHED

The study of asymptotic behaviour and of infinitesimal objects represents a substantial part of Arab mathematical research. Activated by new disciplines, whose development was itself dependent on that of algebra, and particularly of numerical analysis and of the theory of algebraic equations, we meet it in the exposition of approximation methods or in research on maxima, as we have seen in the preceding chapter. But, independently of algebra and disciplines which are associated with it, this study covers also the time of attempts to understand better and to establish ancient geometry theorems, or to answer new questions raised by the application of geometry. Let us mention as an example al-Sijzī's writing about the asymptote to an equilateral hyperbola,[1] or that of Ibn Qurra on the deceleration and acceleration of the apparent movement of a moving body on the ecliptic.[2] We could multiply the circumstances where Arab geometers have undertaken this study—the discussion of the famous proposition X–1 from *Elements* is just an example.[3]

But even more important is the research undertaken by the geometers since the ninth century in this area, along three strands of Greek mathematics. The first concerns the calculation of infinitesimal areas and volumes. We shall show how the Arab neo-Archimedeans advanced the research of the Syracuse mathematician. The second strand concerns the quadrature of lunules; we shall see that for this subject Ibn al-Haytham places himself closer to Euler than to Hippocrates of Chios. Finally, the third strand is dedicated to *extrema* areas and volumes, in the course of examination of the isoperimetric problem. These three avenues of mathematical research are the most advanced in the period that we examine here.

THE CALCULATION OF INFINITESIMAL AREAS AND VOLUMES

The calculation of curved areas and volumes, i.e. limited, at least in part, by curved lines, interested Arab mathematicians relatively early. This advanced sector of mathematical research saw daylight in the ninth century, almost at the same time as the translation of three Greek texts in this area, i.e. on what will later be called the exhaustion method, on the study of the area and the volume of curved surfaces and solids, and on the centres of gravity of certain figures.

At the beginning of the ninth century, al-Ḥajjāj ibn Maṭar had translated the *Elements* from Euclid. It is in book X of this work that the mathematicians met the famous fundamental proposition for this calculation, which is as follows: 'Consider two unequal magnitudes. If we take from the larger one a part bigger than its half, if we take from the remainder a part bigger than its half, and if we do this over and over again, a certain magnitude will remain which will be smaller than the smallest of the proposed magnitudes.'[4] In other words:

Let a and b be two given magnitudes, $a>0$ and $b>0$, such that $a<b$; and let $(b_n)_{n\geq 1}$ be one series such that for all n we have

$$b_n > \frac{1}{2}\left(b - \sum_{k=1}^{n-1} b_k\right)$$

Then there exists n_0 such that, for $n>n_0$, we have

$$b - \sum_{k=1}^{n} b_k < a$$

Two works of Archimedes were also translated into Arabic: *The Measurement of the Circle* (Κύϰλου μέτϱησις) and *The Sphere and the Cylinder* (Πεϱί σφαιϱας ϰαί ϰυλίνδϱου). The translation of the first book was known by al-Kindī and the Banū Mūsā,[5] while the translation of the second book had been revised by their collaborator Thābit ibn Qurra. As for the other books of Archimedes, *On Spirals, On Conoids and Spheroids, On the Quadrature of the Parabola,* and *On the Method,* there is no indication that they were known by the Arab mathematicians. This remark is all the more important since Archimedes introduced in his book *The Conoids and the Spheroids* the notion of inferior and superior integral sums, which would then complete the exhaustion method. The translation of the two treatises of Archimedes as well as the commentary of Eutocius (these texts were translated twice during the ninth century)[6] were an answer to the demand of al-Kindī, the Banū Mūsā and their school. The Banū Mūsā were three brothers, Muḥammad, Aḥmad and al-Ḥasan, who dedicated themselves not only to geometry—notably to conic sections—but also to mechanics, music and astronomy. These three brothers wrote the first Arabic essay on this subject in Baghdad during the first half of the ninth century. Their treatise entitled *On the Measurement of Plane and Spherical Figures* not only launched Arabic research on the determination of areas and volumes, but also remained the fundamental text for Latin science after its translation in the twelfth century by Gerard of Cremona. The treatise is divided into three parts. The first concerns the measurement of a circle, the second the volume of a sphere, and the third treats the classical problems of two means and the trisection of an angle.

In the first part the Banū Mūsā determine the area of a circle by an indirect application of the exhaustion method. They seem to use implicitly a proposition from book XII of the *Elements,* namely: 'For two concentric circles, draw in the larger one a polygon whose equal and even-numbered sides do not touch the smaller circle/ They then demonstrate the following proposition.

Given a segment and a circle, if the length of the segment is less than the circumference of the circle, then we can inscribe inside the circle a polygon the sum of whose sides is greater than the length of the given segment, and if the length of the segment is greater than the circumference of the circle, then we can circumscribe the circle with a polygon the sum of whose sides is less than the length of the given segment.

The Banū Mūsā show next that the area S of the circle is equal to $rc/2$ (where r is the radius and c the circumference). But in this demonstration they do not compare S with $S'>S$, and then with $S''<S$, but they assume that $S=rc/2$ and compare c with a $c'>c$ and with a $c''<c$, contenting themselves in this way with comparing lengths.

The Banū Mūsā then explain the method of Archimedes for the approximate calculation of π, and give a general statement. They actually show that this method reduces to the construction of two adjacent series $(a_n)_n\geq 1$ and $(b_n)_n\geq 1$, with $a_n<b_n$ for all n, that converge to the same limit $2r\pi$. We can rewrite the two series as follows:

$a_n=2nr\sin(\pi/n)$ $b_n=2nr\tan(\pi/n)$

They note that 'it is possible with the help of this method to reach any degree of precision'.[7] With an analogous method to the one applied in the case of the area of a circle, they determine the area of the surface of a sphere. Here again, they base themselves indirectly on the same proposition of book XII of the *Elements* of Euclid and on geometrical transformations. Their method here is different from that of Archimedes, even if the fundamental ideas are the same. By this method they show that the area of the surface of a sphere is equal to four times the area of the great circle of the sphere, i.e. $4\pi r^2$. The Banū Mūsā determine the volume of the sphere as 'the product of its half-diameter by a third of the area of the surface', i.e. $\frac{4}{3}\pi r^3$. Let us note finally that, with regard to this part of the treatise and also with regard to the trisection of an angle the Banū Mūsā claim to have done these studies, whereas they attribute the approximate calculation of π to Archimedes and they attribute to Menelaus the determination of two segments among two other given segments so that the four are in continued proportion.

The contemporaries and successors of the Banū Mūsā followed very actively the research in this area. Thus al-Māhānī not only commented on the book of Archimedes on *The Sphere and the Cylinder* but also dealt with the determination of a segment of a parabola. This text from al-Māhānī has not survived to our time.

The Banū Mūsā's collaborator, Thābit ibn Qurra (d. 901) contributed massively in this area. He wrote successively three treatises: one was dedicated to the area of a segment of a parabola, the second to the volume of the paraboloid of revolution, and the third was on the sections of a cylinder and its area.

In the first treatise, to determine the area of a segment of a parabola, Thābit ibn Qurra, who ignored Archimedes' study on this subject, starts by demonstrating twenty-one lemmas, of which fifteen are arithmetical. Examination of these lemmas shows that Thābit ibn Qurra knew perfectly well the concept of the upper limit of a set of real square numbers, and the uniqueness of this upper limit. In fact, Thābit ibn Qurra uses the following property to characterize the upper limit.

Let ABC be a segment of a parabola, AD its diameter corresponding to BC (Figure 13.1). With every given $\varepsilon>0$, we can associate a partition A, G_1, G_2,…, G_{n-1}, D of the diameter AD such that area BAC−area of polygon $BE_{n-1}\ldots E_2E_1AF_1F_2\ldots F_{n-1}C<\varepsilon$

That is, in other words, the area BAC is the upper limit of the areas of these polygons.

Thābit ibn Qurra demonstrates in a similar rigorous way that two-thirds of the area BHMC is the upper limit of the areas of the polygons already mentioned. He finally arrives at his theorem which he states as follows: 'The parabola is infinite but the area of any one of its parts is equal to two-thirds of the parallelogram with the same base and height as the part'.[8] Here is the outline of his demonstration.

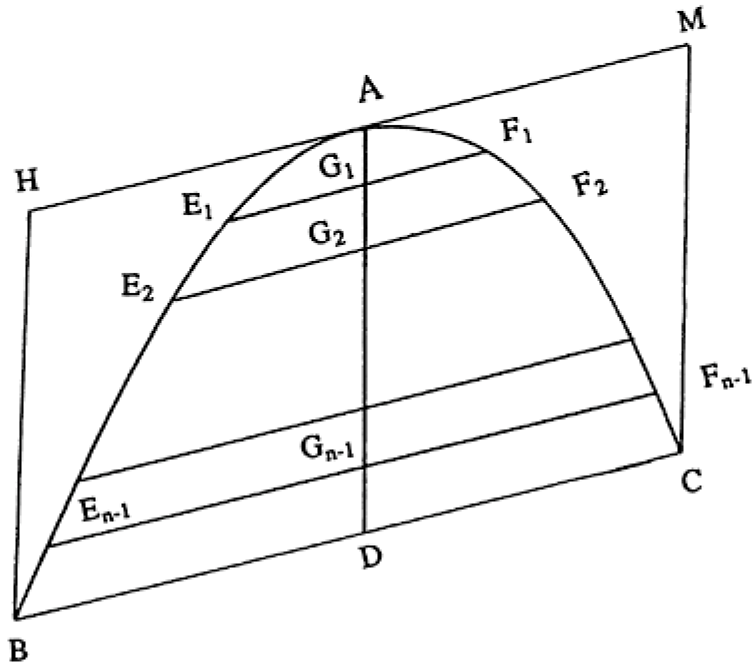


Figure 13.1

Let $\mathscr{S}$ the area of the segment of the parabola *P*, and *S* the area of the parallelogram with the same base and height. If $\mathscr{S}\neq \frac{2}{3}S$, we have two cases:

(a) $\mathscr{S}>\frac{2}{3}S$

Let $\varepsilon>0$ be such that

$$\mathscr{S}-\tfrac{2}{3}S=\varepsilon \qquad (1)$$

From a lemma already demonstrated, for this ε there exists N such that for $n>N$ there exists P_n of area S_n such that

$$\mathscr{S} - S_n < \varepsilon \tag{2}$$

From (1) and (2) we deduce that

$$(\tfrac{2}{3}S + \varepsilon) - S_n < \varepsilon$$

whence

$$\tfrac{2}{3}S < S_n$$

But, from another lemma, we have

$$\tfrac{2}{3}S > S_n$$

which is a contradiction.

(b) $\mathscr{S} < \frac{2}{3}S$

Let $\varepsilon>0$ be such that

$$\tfrac{2}{3}S - \mathscr{S} = \varepsilon \tag{3}$$

From a lemma demonstrated previously, for this ε there exists N such that, for $n>N$, there exists P_n with area S_n such that

$$\tfrac{2}{3}S - S_n < \varepsilon \tag{4}$$

From (3) and (4) we have

$$(\mathscr{S} + \varepsilon) - S_n < \varepsilon$$

whence

$$\mathscr{S} < S_n$$

But P_n is inscribed in *P,* and therefore $S_n < \mathscr{S}$, which is a contradiction.

The exhaustion method applied here by Ibn Qurra rests, as we can see, on properties of the upper limit, and particularly its uniqueness. In effect we want to show that $\frac{2}{3}S = \mathscr{S}$, knowing that

1 $\mathscr{S}$ the upper limit of $(S_n)_{n\geq 1}$

2 $\frac{2}{3}S$ is the upper limit of $(S_n)_{n\geq 1}$

In fact, in Ibn Qurra's approach, we recognize the fundamental idea of the Riemann integral. In the particular case when the diameter considered is the axis of the parabola, Thābit ibn Qurra's approach comes down to taking a partition $\sigma = AG_1G_2 \ldots G_{n-1}$ of diameter AD (cf. Figure 13.2) and then taking the sum

$$S_\sigma = \sum_{i=1}^{n} (AG_i - AG_{i-1}) \frac{G_{i-1}F_{i-1} + G_iF_i}{2}$$

and showing that

$$\forall \varepsilon > 0 \; \exists \, \sigma \text{ such that the area ACD} - S_\sigma < \varepsilon$$

Finally he proves in other words that S_σ converges to this area following the filter of the partitions of AD.

Let us translate the preceding in the language of analysis. Let x_i be the abscissa of G_i and let $y=f(x)$ be the equation of the parabola. S_σ can then be written

$$S_\sigma = \sum_{i=1}^{n} (x_i - x_{i-1}) \frac{f(x_{i-1}) + f(x_i)}{2}$$

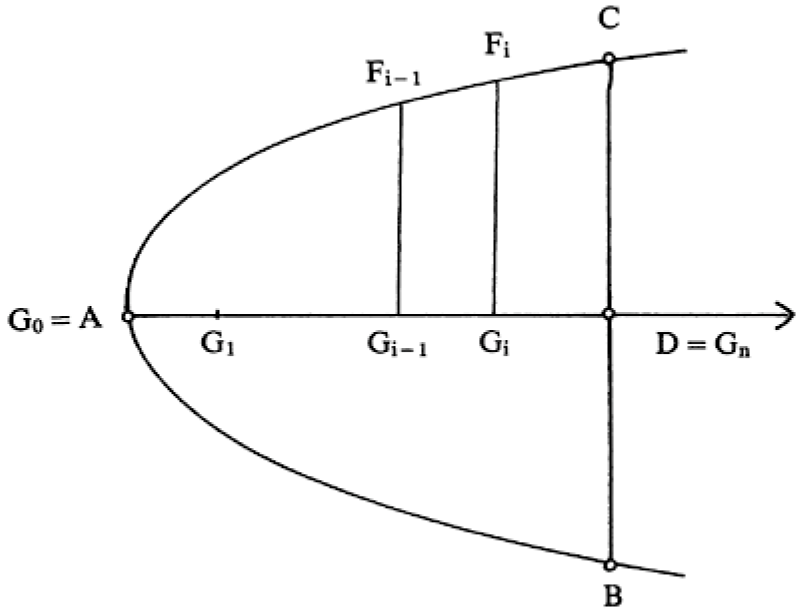


Figure 13.2

but since

$$f(x_{i-1}) \leqslant \frac{f(x_{i-1}) + f(x_i)}{2} \leqslant f(x_i)$$

and since *f* is continuous, we deduce that

$$\frac{f(x_{i-1}) + f(x_i)}{2}$$

is a value reached by *f* at the point ξ_i of the interval $[x_{i-1}, x_i]$. S_σ can then be written in the form

$$S_\sigma = \sum_{i=1}^{n} (x_i - x_{i-1}) f(\xi_i) \qquad x_{i-1} \leqslant \xi_i \leqslant x_i$$

which is just the sum used in the definition of the Riemann integral of a function *f*. Finally let us note that the quadrature of Ibn Qurra, given the definition of the parabola, is equivalent to calculation of the integral $\int_0^a \sqrt{(px\,\mathrm{d}x)}$. As a modern historian, M.A.Youschkevitch, writes about the procedure of Thābit ibn Qurra:

> Thanks to this procedure, Ibn Qurra revived the method, almost forgotten, of the calculation of integral sums. In addition, with the help of this same procedure, Ibn Qurra has effectively calculated, for the first time, an integral $\int_0^a x^n\, dx$ for a fractional value of the exponent *n,* i.e. $\int_0^a x^{1/2}\, dx.$ This done, he proceeds again for the first time to a subdivision of the integration interval into unequal parts. This is done by an analogous procedure consisting of dividing the axis of the abscissae into segments forming a geometrical series, by P.Fermat, in the middle of the seventeenth century, undertakes the quadrature of the curves $y=x^{m/n}$, with $(m/n)\neq 1$.[9]

The contribution of Ibn Qurra to this subject does not stop here. He undertakes the determination of the volume of a paraboloid of revolution. Here also, the study begins with a large number of lemmas—thirty-five. To determine this volume, Ibn Qurra uses frustums of adjacent cones, the bases of which determine a subdivision of the diameter of the parabola—which generates the paraboloid—the intervals of which are proportional to successive odd numbers starting with one, and whose heights are the same.

Thābit ibn Qurra finally undertakes—as al-Ḥasan ibn Mūsā before (Rashed 1995: chap. I)—in a treatise on *The Sections of a Cylinder and their Surface* the study of different kinds of plane sections of a straight cylinder and an oblique cylinder; he determines then the area of an ellipse and the area of elliptic segments, discusses the maximal and minimal sections of the cylinder and their axes, and determines the area of part of a surface bounded by two plane sections.

It is impossible to show here the results and the proofs of this rich and profound treatise, like the demonstration by which Thābit ibn Qurra shows that 'the area of an ellipse is equal to the area of the circle whose semi-diameter squared is equal to the product of one of the two axes of this ellipse by the other', i.e. πab, with a and b the half-axes of the ellipse.

Thus, with Thābit ibn Qurra, research on infinitesimal determinations is already quite advanced, and his successors will then try to develop his achievements; this is the case in particular with the grandson of Thābit ibn Qurra, Ibrāhīm ibn Sinān, al-Qūhī, Ibn Sahl and Ibn al-Haytham.

We have already seen that Thābit ibn Qurra reintroduced the concept of integral sums. This concept is present with Archimedes, certainly, but in the treatises not translated into Arabic. It remains that a profound study of the two treatises translated into Arabic placed on the path of this rediscovery a mathematician of the stature of Ibn Qurra. Even better, the integral sums of Thābit are more general than those of Archimedes in the sense that Thābit has taken as subdivisions intervals where the steps are not necessarily equal. As for his study of the paraboloid, where he has always proceeded by integral sums, he did not consider, like Archimedes, cylinders of equal heights, but a cone and frustums of cones having the same height, and whose bases are in the proportion of successive odd numbers beginning with one.

The contribution of Ibn Qurra, we have already said, will be actively followed by his successors, such as his grandson Ibrāhīm ibn Sinān. This mathematician of genius lived only thirty-seven years, and in his own words was unhappy that 'al-Māhānī has a study more advanced than that of [my] grandfather, without any of us managing to go further than him'. He wants to give a demonstration not only shorter than his grandfather's, which needed twenty lemmas, as we have seen, but also shorter than al-Māhānī's. The proposition on which Ibrāhīm ibn Sinān's demonstration is founded, and which he had demonstrated before, is: *the affine transformation leaves the proportionality of the areas invariant.*

Ibn Sinān's method comes down to considering the polygon as a sum of 2^n-1 triangles, inscribed in the area of a parabola. Let a_1 be the triangle EOE′, a_2 the polygon ECOC′E′, and so on (Figure 13.3). Ibn Sinān demonstrates that, if a_n and a'_n are two polygons drawn respectively in the two areas a and a' of a parabola, then

$$\frac{a_n}{a'_n} = \frac{a_1}{a'_1} \qquad n = 1, 2, \ldots$$

He actually demonstrates an expression equivalent to

$$\frac{a}{a'} = \lim_{n\to\infty} \frac{a_n}{a'_n} = \frac{a_1}{a'_1}$$

from where he deduces

$$\frac{1}{2}\cdot\frac{a-a_1}{a} = \frac{1}{2}\cdot\frac{a_2-a_1}{a_1} = \frac{1}{8}$$

and finally obtains

$$a = \frac{4}{3}\, a_1$$

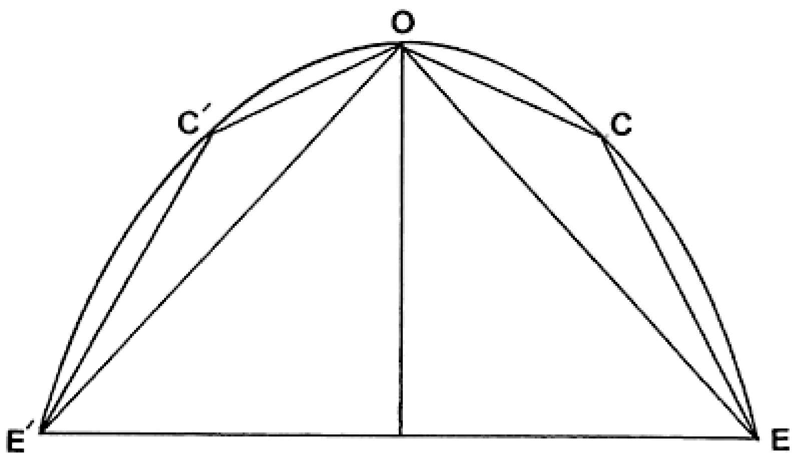


Figure 13.3

We note that it was the introduction of the affine transformation which allowed the number of lemmas necessary to be reduced to two.

In the tenth century the mathematician al-ʿAlāʾ ibn Sahl[10] took up again the quadrature of the parabola, but his treatise has unfortunately not been found. His contemporary al-Qūhī, when re-examining the determination of the volume of the paraboloid of revolution, rediscovered Archimedes' method. For the paraboloid of revolution, Archimedes considers cylinders of the same height, while Thābit ibn Qurra used, as we have seen, adjacent frustums of a cone whose bases determine a subdivision of the diameter of a parabola—which generates the paraboloid—whose intervals are proportional to successive odd numbers starting with one, and whose heights are equal. Al-Qūhī,[11] to achieve a reduction, as he claims, from thirty-five to two in the number of lemmas used by Thābit ibn Qurra, independently found the integral sums that occur in Archimedes' work. It is only in some detailed points that his method is different from Archimedes' method, in particular in the proof that one can make the difference between inscribed cylinders and circumscribed cylinders as small as one wants.

A successor of Ibn Sahl and of al-Qūhī,[12] the famous mathematician and physicist Ibn al-Haytham (d. after 1040) takes up again the demonstration of the volume of the paraboloid of revolution, as well as that of the volume generated by the rotation of a parabola around its ordinate. Let us quickly consider this second case, which is more difficult than the first. To determine this volume, Ibn al-Haytham starts by demonstrating some arithmetical lemmas—the sums of the powers of n successive integers—in order to establish a double inequality, fundamental for his study. On this occasion he obtains results that are a landmark in the history of arithmetic, in particular the sum of any integer power of n successive prime integers

$$\sum_{k=1}^{n} k^i \qquad i = 1, 2, \ldots$$

He establishes then the following inequality:

$$\sum_{k=1}^{n} [(n+1)^2 - k^2]^2 \leqslant \frac{8}{15}(n+1)(n+1)^4 \leqslant \sum_{k=0}^{n} [(n+1)^2 - k^2]^2 \qquad (1)$$

Now consider the paraboloid generated by the rotation of a segment of the parabola ABC with equation $x=ky^2$ around the ordinate BC (Figure 13.4). Let $\sigma_n = (y_i)_{0 \leqslant i \leqslant 2^m}$, with $2^m=n$ a subdivision of the interval $[0, b]$ of the step

$$h = \frac{b}{2^m} = \frac{b}{n}$$

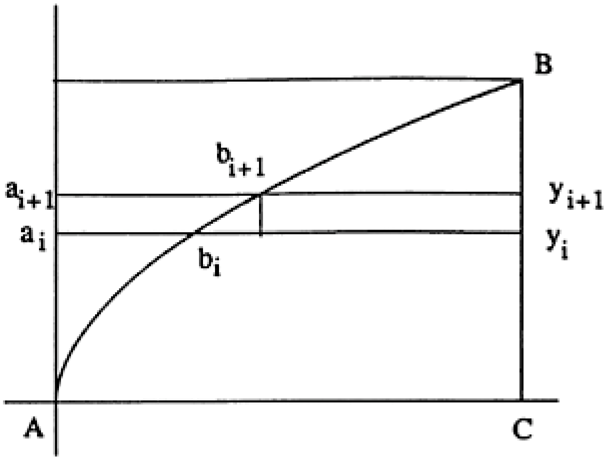


Figure 13.4

Let M_i be the points of the parabola of ordinates y_i and abscissae x_i respectively. Put

$r_i = c - x_i \quad 0 \le i \le 2^m = n$

We obtain

$$r_i = k(b^2 - y_i^2) = kh^2(n^2 - i^2)$$

We have

$$I_n = \sum_{i=1}^{n-1} \pi k^2 h^5 (n^2 - i^2)^2$$

and

$$C_n = \sum_{i=0}^{n-1} \pi k^2 h^5 (n^2 - i^2)^2$$

But, from inequality (1), we obtain

$$I_n \leqslant \frac{8}{15} V \leqslant C_n$$

where $V = \pi k^2 b^4$. b is the volume of the circumscribed cylinder.

Using a different language from that of Ibn al-Haytham: as the function $g(y)=ky^2$ is continuous on [0, *b*], the calculation of Ibn al-Haytham will be equivalent to the following.

Volume of the paraboloid

$$v(p)=\lim_{n\to\infty}\sum_{i=1}^{n}\pi k^2h^5(n^2-i^2)^2$$

whence

$$v(p)=\lim_{n\to\infty}\sum_{i=1}^{n}\pi k^2(b^4-2b^2y_i^2+y_i^4)h$$

whence

$$v(p)=\pi\int_0^b k^2(b^4-2b^2y^2+y^4)\,dy$$

whence

$$v(p)=\frac{8}{15}\pi k^2b^5=\frac{8}{15}V$$

where *V* is the volume of the circumscribed cylinder.

Ibn al-Haytham does not stop here: he turns again to the small enclosing solids, in order to study their behaviour when the subdivision points are increased indefinitely. We find ourselves this time in the presence of a frankly infinitesimalist thought, and in some sort functional, in the sense that the heart of the problem is explicitly the asymptotic behaviour of mathematical entities for which we try to determine the variation.

Ibn al-Haytham applies the same method to the determination of the volume of a sphere. Here also, we note that he has given an arithmetically adapted version of the exhaustion method. In his research, the role of explicit arithmetical calculation seems to be much more important than in the work of his predecessors. But let us consider for the moment his method from the point of view of integral calculation, in order to bring out its basic ideas.

To determine the volumes of revolution around a given axis, Ibn al-Haytham, we have seen, takes cylindrical sections inscribed and circumscribed, whose axis is that of the solid of revolution considered. This allows approximations greater than and less than the volume to be calculated by integral sums—Darboux sums—relative to the function which corresponds to the curve generating the solid of revolution considered. For the volume of a sphere, for instance, he considers

$$I_n = \sum_{i=1}^{n-1} \pi y_i^2 (x_i - x_{i-1}) = D(f, \sigma_n, m_i)$$

$$C_n = \sum_{i=1}^{n} \pi y_{i-1}^2 (x_i - x_{i-1}) = D(f, \sigma_n, M_i)$$

Note that the function f is monotonic, so that m_i and M_i are the values of f at the limits of the ith interval of the subdivision, f being the function defined by (see Figure 13.5)

$$f(x) = \pi(R^2 - x^2) = \pi y^2 \qquad m_i = \inf_{x_{i-1} \leqslant x \leqslant x_i} f(x) = y_i \qquad \text{and}$$

$$M_i = \sup_{x_{i-1} \leqslant x \leqslant x_i} f(x) = y$$

However, Ibn al-Haytham uses next the inequalities

$$I_n < v < C_n$$

and demonstrates that, for all $\varepsilon > 0$, there exists N such that for $n \geq N$ we have

$$v - I_n < \varepsilon \qquad C_n - v < \varepsilon$$

which proves that I_n tends to v and the same for C_n; i.e.

$$v = \int_0^R f(x)\, \mathrm{d}x$$

In other terms, the calculation of Ibn al-Haytham is equivalent to that of a simple Cauchy-Riemann integral.

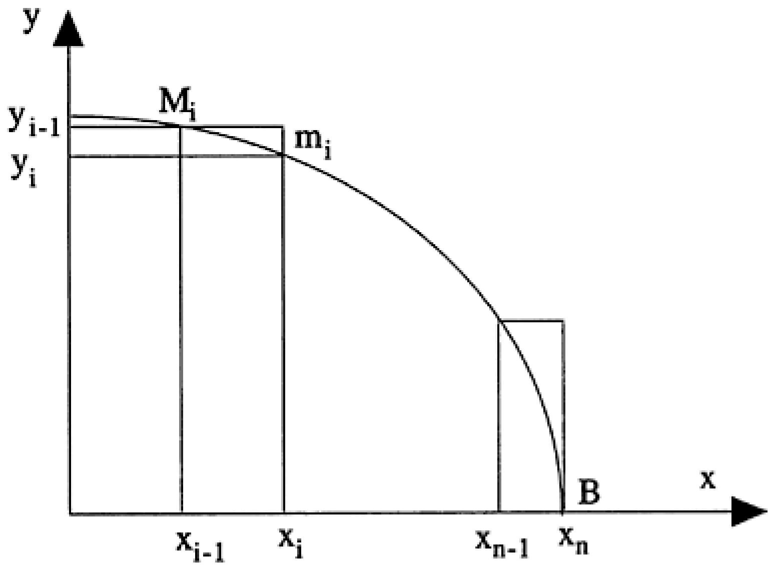


Figure 13.5

But this mathematical equivalence should not hide the question: why did Ibn al-Haytham, after having determined these volumes with the help of this integral, never explicitly sketch a general method to determine other volumes and other areas? We cannot answer this question in a satisfactory way by just evoking the requirements of Ibn al-Haytham—it is true that, in his mathematical work, optical or astronomical, there was no need for calculation of the volume of a paraboloid, or for calculation of the volume of a hyperboloid of revolution for example. It is thus the method itself that caused the absence of such a sketch.

We note that Ibn al-Haytham—as well as his predecessors in the case of areas—always had recourse to another solid of known volume with which he could compare the considered solid. This preliminary knowledge of a comparable solid is by no means an artifact of the method: it gives Ibn al-Haytham as well as his predecessors an effective calculation—direct and exact—of the limits of the corresponding Darboux sums. In the general case, comparable solids do not necessarily exist, which makes the mathematical tools adopted by Ibn al-Haytham insufficient for an effective calculation of the Darboux sums. It is therefore an internal limitation which marks Ibn al-Haytham's method. However, we should be careful not to exaggerate the impact of this limitation, which will disappear when an even more arithmetical calculation is introduced. Thus although the reference volume identifies the Archimedean tradition, the arithmetical reorientation which was growing in the Arabic tradition shows that it was not the Archimedean inheritance any more. It is not only geometry any more that guides the steps of Ibn al-Haytham, but arithmetic, and the lemmas were conceived in an arithmetic perspective of the figures.

In this study we can already see the development of the means and techniques of this area of Arabic mathematics. We have seen that Ibn al-Haytham, in his research on the paraboloid, has obtained results that the historians attribute to Kepler and Cavalieri for instance. This subject stops here, however, and this is very probably because of the absence of an efficient symbolism.

THE QUADRATURE OF LUNULES

Amongst the problems of determining the areas of curved surfaces, the exact quadrature of lunules—surfaces limited by two arcs of circles—is one of the most ancient. According to later testimonies—Simplicius, Aristotle's commentator in the sixth century—this problem goes back to Hippocrates of Chios, i.e. to the fifth century before our time. In his commentary about the *Physics* of Aristotle, Simplicius[13] mentions a long passage by Eudemus, Aristotle's student, which contains the results and methods of Hippocrates. This passage, which raises several philological and historical problems which we cannot deal with here, is the only known source for the history of this problem in Greek mathematics. It also indicates the context where this problem regarding the quadrature of certain lunules was posed, i.e. the quadrature of a circle.

About five centuries after Simplicius, Ibn al-Haytham comes several times to the same problem, first in relation to the quadrature of a circle, and then just for the problem itself. He deals with it in three dissertations of which only one has been studied until now—his dissertation on the quadrature of a circle—and he dedicates a short dissertation to the quadrature of lunules. Later he regards this question again to obtain results attributed to seventeenth- and eighteenth-century mathematicians. The ignorance about Ibn al-Haytham's works, and in particular this later treatise, has led historians, in all good faith, to make erroneous judgements about his contribution to this research.

It seems that the starting point of Ibn al-Haytham can be found in the text attributed to Hippocrates of Chios. Thus, in his first treatise, he starts by writing: 'When I examined…, the figure of the lunule equal to a triangle, mentioned by the Ancients…,'.[14] Moreover the results of Hippocrates of Chios are integrated in the works of Ibn al-Haytham. Did he know of them through the commentary on the *Physics* of Aristotle by Simplicius, which by then had most likely been translated into Arabic? We do not have documents that allow us to answer this question.[15] Whatever may be the case, let us go to the two treatises of Ibn al-Haytham.

In these two treatises, Ibn al-Haytham studies lunules limited by any arcs, looking for equivalences in surfaces. He introduces circles equivalent in general to given sectors of the circle in the problem, and expressed as a fraction of it. He justifies the existence of the circles introduced, which he must add to or remove from the polygonal surfaces, to obtain a surface equivalent to that of a lunule or a sum of two lunules. In the first treatise, succinct, he starts, in the three propositions 1, 2 and 5, from a semicircle ABC to study the lunules L_1 and L_2 limited by an arc AB or BC, and by a semicircle (Figure 13.6). He assumes that arc AB is equal to a sixth of the circumference and establishes the following results.

$$L_1 + \frac{1}{24}\,\mathscr{C}(\text{ABC}) = \frac{1}{2}\,\text{tr(ABC)}$$

$$L_2 = \frac{1}{2}\,\text{tr(ABC)} + \frac{1}{24}\,\mathscr{C}(\text{ABC})$$

$$L_2 + \frac{1}{2}\,\text{tr(ABC)} = L_3 + \frac{1}{8}\,\mathscr{C}(\text{ABC})$$

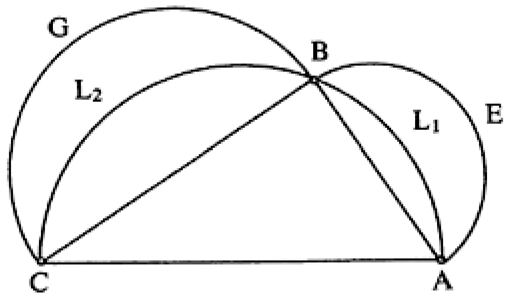


Figure 13.6

where L_3 is a lunule similar to L_1 such that $L_3=2L_1$; $\mathscr{C}(\text{ABC})$ and tr(ABC) designate respectively the circle ABC and the triangle ABC.

In proposition 3 of this treatise, Ibn al-Haytham slightly generalizes the demonstration of the result of Hippocrates of Chios by considering any point B on the semicircle ABC:

$L_1+L_2=\text{tr(ABC)}$;

and in proposition 4 he studies the ratio of two similar lunules.

We note that, in these propositions, the lunules L_1 and L_2 mentioned are lunules associated with the three semicircles ABC, AEB and BGC.

The first treatise of Ibn al-Haytham is on the same lines of research as that of Hippocrates of Chios. It is the same for the part relating to the lunules in his treatise on *The Quadrature of the Circle*.[16] We see that Ibn al-Haytham, like Hippocrates of Chios, uses the proportionality of the circle's area to the square of the diameter, and Pythagoras's theorem. In both cases the lunule associated with a right-angled isosceles triangle is studied. Even if the reasoning of Ibn al-Haytham is slightly more general, this generality does not profoundly alter the similarity of his approach with that of Hippocrates of Chios. Let us note that the important thing in his treatise on *The Quadrature of the Circle* is not the results on the lunules that he studied (as in his first treatise) but the clear difference that he established between the existence of a square equivalent to a circle—i.e. for us the existence of a transcendent ratio—and the constructability of this square or of this ratio.[17]

The situation is profoundly modified in his second treatise.[18] Not only does Ibn al-Haytham obtain more general results but his approach is not the same: he takes anew the problem of the quadrature of lunules at a fundamental level, moves it to a trigonometry plane, and attempts to deduce the different cases as many of the properties of a trigonometric function which will be more precisely recognized much later by Euler.

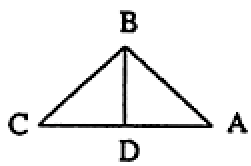
From the beginning of this treatise, Ibn al-Haytham explicitly recognizes that the calculation of the areas of lunules involves sums and differences of the areas of sectors of circles, and of the triangles whose comparison demands in its turn a comparison of the ratios of angles and the ratios of segments. It is for this reason that he starts by establishing four lemmas with regard to triangle ABC, right-angled at B in the first lemma and with an obtuse angle in the three others, which show from now on that the essential point of the study is connected with the study of the function

$$f(x)=\frac{\sin^2 x}{x} \qquad 0<x\leqslant\pi \tag{1}$$

We can therefore rewrite these lemmas as follows

1 If

$$0<C<\frac{\pi}{4}<A<\frac{\pi}{2}$$



then

$$\frac{\sin^2 C}{C}<\frac{2}{\pi}<\frac{\sin^2 A}{A}$$

It is obvious that if $C=A=\pi/4$, then

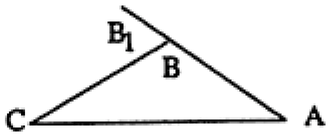
$$\frac{\sin^2 C}{C}=\frac{\sin^2 A}{A}=\frac{2}{\pi}$$

2 Let $\pi-B=B_1$. If

$$C<\frac{\pi}{4}<B_1<\frac{\pi}{2}$$

then

$$\frac{\sin^2 C}{C} < \frac{\sin^2 B_1}{B_1}$$



3 If $A \leq \pi/4$, then

$$\frac{\sin^2 A}{A} < \frac{\sin^2 B_1}{B_1}$$

4 Here Ibn al-Haytham wants to study the case $A > \pi/4$; but the study is incomplete. He shows that for a given A we can find B_0 such that

$$B_1 \geqslant B_0 \Rightarrow \frac{\sin^2 A}{A} > \frac{\sin^2 B_1}{B_1}$$

This incomplete study seems to have hidden for Ibn al-Haytham the equality

$$\frac{\sin^2 A}{A} = \frac{\sin^2 B_1}{B_1}$$

We note that these lemmas, because they connect the problem of the quadrature of lunules with trigonometry, change the situation and allow consolidation of particular cases. But the incompleteness already mentioned has disguised the possibility of the existence of quadrable lunules. Let us turn for the moment, briefly, to the propositions in the second treatise of Ibn al-Haytham.

In nine propositions −8 to 16—the lemmas are associated in pairs, and in all cases the three arcs ABC, AEB, BGC are similar. Let O, O_1 and O_2 be the centres of the corresponding circles (Figure 13.7); put

$$\angle AOC = \angle AO_1B = \angle BO_2C = 2\alpha \qquad \angle AOB = 2\beta \qquad \text{and} \qquad \angle BOC = 2\beta'$$

with $\beta \leq \beta'$ and $\beta + \beta' = \alpha$.

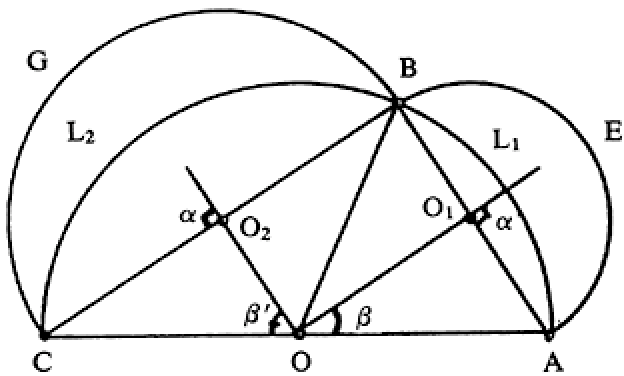


Figure 13.7

Lunule L_1 is characterized by *(α, β)* and lunule L_2 by *(α, β')*. We then consider the case $a=\pi/2$; we have the following propositions.

1 For all *(β, β')* such that $\beta+\beta'=\pi/2$, we have L_1+L_2=tr(ABC).

2 For $\beta=\beta'=\pi/4$, we have $L_1 + L_2 = \frac{1}{2}\text{tr(ABC)};$ in this case we have $\alpha/\beta = \frac{2}{1}$ and the only quadrable lunule studied by Ibn al-Haytham.

For $\beta<\beta'$ we have

$$L_1 = \tfrac{1}{2}\text{tr(ABC)} - \mathscr{C}(\text{N})$$
$$L_2 = \tfrac{1}{2}\text{tr(ABC)} + \mathscr{C}(\text{N})$$

The circle (N) depends on the ratio α/β.

3 For $\beta=\pi/6$, we have $L_1 = \frac{1}{2}\text{tr(ABC)} - \frac{1}{24}\mathscr{C}(\text{ABC});$ in this case, we have $\alpha/\beta = \frac{3}{1}$.

For $\beta'=\pi/3$, we have $L_2 = \frac{1}{2}\text{tr(ABC)} + \frac{1}{24}\mathscr{C}(\text{ABC});$ in this case we have $\alpha/\beta' = \frac{3}{2}$.

Until now Ibn al-Haytham had only used lemma 1 in his demonstrations; to establish the following proposition, he uses the three other lemmas. The directing idea is to start from the points M and N on the chord AC such that

<ABC=<BMC=<ANB=$\pi-\alpha$

and define a point P on AB and a point Q on BC such that NP//OA and MQ//OC (Figure 13.8); indeed the results cannot be established from the triangle ABC as in the previous propositions.

Thus, for all pairs *(β, β')* such that $\beta+\beta'<\pi/2$, Ibn al-Haytham defines two circles (K) and (Z) such that

$L_1+L_2+(K)$=quadrilateral (OPBQ)
$L_1+(Z)=\text{tr}(OPB)$

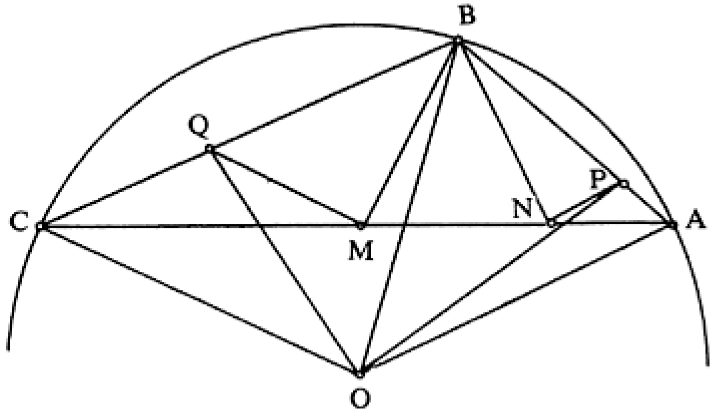


Figure 13.8

He then examines the following cases:
If $\beta=\beta'$, we have $(Z)=1/2\ (K)$, $L_1=L_2$, $L_2+(Z)=\text{tr}(OQB)=\text{tr}(OPB)$.
If $\beta'<\pi/4$, we have $(Z)<(K)$, $L_2+(K)-(Z)=\text{tr}(OQB)$.
If $\beta'>\pi/4$, we can have

$(Z)<(K)$, $L_2+(K)-(Z)=\text{tr}(OQB)$ and $L_2<\text{tr}(OQB)$

or

$(Z)=(K)$, $L_2=\text{tr}(OQB)$

or

$(Z)>(K)$, $L_2=\text{tr}(OQB)+(Z)-(K)$, $L_2>\text{tr}(OQB)$

Ibn al-Haytham next illustrates these results by examples and then demonstrates these propositions:

4 If $\alpha=\pi/3$, $\beta=\beta'=\pi/6$, $\alpha/\beta=2/1$, we have

$$L_1 = L_2 = \tfrac{2}{3}\text{tr(ABC)} - \tfrac{1}{18}\mathscr{C}\text{(ABC)}$$

5 If $\alpha=\pi/3$, $\beta=\pi/12$, $\beta'=\pi/4$, $\alpha/\beta=\frac{4}{1}$, $\alpha/\beta'=\frac{4}{3}$, in this case the circle involved is not a fraction of the circle (ABC).

6 If $\alpha=\pi/3+\pi/8$, $\beta=\pi/8$, $\beta'=\pi/3$, $\alpha/\beta=\frac{11}{3}$, $\alpha/\beta'=\frac{11}{8}$, in this case the circle involved is not a fraction of the circle (ABC).

In the following propositions, with the exception of proposition 21, Ibn al-Haytham studies the figures formed by sums or differences of lunules and segments of triangles. In proposition 21, he indicates a property of the lunule whose two arcs belong to two equal circles. This property results from the translation which associates two circles; the property is studied by Ibn al-Haytham in his treatise on *Analysis and Synthesis*.[19]

With the second treatise of Ibn al-Haytham, the study of the quadrature of lunules is undertaken in another direction which will later lead to Euler, shifting the problem towards trigonometry and recognizing in some way its dependence on function (1).

THE ISOPERIMETRIC PROBLEM

The problem is to show that in a plane, for a given perimeter the disc has the largest area; and that, of all the solids in space with the same surface area, the sphere has the largest volume. From later evidence,[20] this seems to have been an old result. Zenodorus dealt with the problem, and having established the demonstration, in his lost treatise *On the Isoperimetric Figures* (Περί ἰσομέτρων σχημάτων).[21] But for mathematical as well as cosmographical reasons, this problem continued to interest mathematicians, astronomers and even philosophers. Let us mention, amongst others, Hero of Alexandria,[22] Ptolemy,[23] Pappus,[24] and Theon of Alexandria, but it is Ptolemy and Theon whom we are most interested in here. In the *Almagest,* with the help of his thesis about sphericity, so important for his astronomy and his cosmogony, Ptolemy recalls the preceding result as established and writes: ‘Since, among the different but isoperimetric figures, those with more sides are the largest, among plane figures it is the circle which is the largest, and among the solids, the sphere’.[25] As for Theon of Alexandria, he summarizes the book of Zenodorus in his commentary on the first book of the *Almagest;* in his words, after he has posed the problem, ‘we are going to prove it in a short way, from the demonstration of Zenodorus in his treatise about isoperimetric figures’.[26] But already in the first decades of the ninth century the *Almagest* as well as the *Commentary* of Theon of Alexandria were already translated into Arabic.

These were the sources of al-Kindī, who seems to have been the first to deal with this problem in Arabic. He mentions it in his *Great Art* (*Fī al- Ṣināʿat al-ʿUẓma*), where we can clearly find the influence of Theon.[27] Thus, after mentioning the result, he writes: ‘we have already explained it in our book on sphericals’.[28] But the tenth-century biobibliographer al-Nadīm[29] tells us also that al-Kindī dedicated a treatise to this problem, entitled: *The sphere is the largest of the solid figures and the circle is the largest of all plane figures*.

But these writings from al-Kindī have not yet been found, and we cannot say anything about his contribution. It is not possible either to mention research concerning this problem during his and his successors’ time, as the commentary to Ptolemy’s first book by the philosopher and mathematician al-Fārābī is missing. The first substantial study about this problem which has reached us is that of the mathematician from the middle of the tenth century: al-Khāzin.[30]

The study of al-Khāzin, and his successors, as we shall see, seems to take cosmography as its leitmotif. The book starts directly with the quotation from Ptolemy which we have just given, followed by nine lemmas which in themselves show that, although al-Khāzin knew Zenodorus's results in Theon's résumé, he followed another demonstrative path. Let us look briefly at al-Khāzin's exposition.

The first four lemmas of al-Khāzin intend to establish that the area of an equilateral triangle is greater than the area of any isosceles triangle with the same perimeter. In the fifth lemma, he shows that the area of an equilateral triangle is greater than the area of any triangle with the same perimeter. In the sixth lemma he moves on to parallelograms and rhombuses, and compares their areas with that of a square with the same perimeter. In the seventh lemma he takes examples of a pentagon and shows that the area of a regular pentagon is greater than that of a non-regular pentagon having the same perimeter.

If we compare with Zenodorus, we can see the difference between their methods. Zenodorus starts by comparing any triangle with an isosceles triangle having a common base and the same perimeter, arriving at the lemma: 'The sum of two isosceles triangles, similar between themselves but having unequal bases, is greater than the sum of two isosceles triangles not similar between themselves but isoperimetric to similar triangles'. By 'isoperimetric' we should understand here that the sums of the sides are equal, the bases being excluded. But this lemma from Zenodorus is not exact,[31] and it is quite surprising that neither Pappus nor Theon had noted his error. Was it at the root of the choice of method made by al-Khāzin?

Al-Khāzin demonstrates next that, if two regular polygons P_1 and P_2 have sides n_1 and n_2 respectively, with n_1 greater than n_2, and the same perimeter, then the area of P_1 is greater than the area of P_2.

It is then that he shows the 'extreme' property of the circle: if a circle and a regular polygon have the same perimeter, then the area of the circle is greater than the area of the polygon. We can see then that the approach of al-Khāzin is organized as follows: (1) he starts by comparing regular polygons with the same perimeter and different numbers of sides; (2) next he compares a regular polygon circumscribed on a circle and a circle with the same perimeter. This approach, common to al-Khāzin and Zenodorus, is static in the sense that we have on one side the given polygon and on the other the circle.

Let us come now to the second part of al-Khāzin's treatise, dedicated to the isepiphanies. Here again, after stating several lemmas about the area of a pyramid and its volume, the area of a cone, of a frustum of a cone and its volume, he ends up by establishing three fundamental propositions. The first can be written as follows.

> Let Σ be a solid of revolution formed by frustums of cones and of cones inscribed in a sphere S of radius R; and let S′ be a sphere of radius R' inscribed in Σ; we show that $4\pi R^2 < \text{area } \Sigma < 4\pi R'^2$.

In the second proposition he shows that the area of a sphere is equal to four times the area of its great circle. In the third he determines the volume of a sphere. In order to arrive at this, al-Khāzin defines a particular solid inscribed in the sphere, and admits the existence of a sphere tangential to all the faces of the solid; which is inexact. The result obtained, however, is exact. Then finally he demonstrates the 'extreme' property of the sphere in the following way.

> Consider a sphere with centre O and radius R, of area S and volume V, and a polyhedron of the same area S and with volume V_1 which is circumscribed on another sphere of radius R', of area S'; we then have
>
> $$V_1 = \frac{1}{3} S.R'$$
>
> The area S' is less than the area of the polyhedron, i.e. $S > S'$ and consequently
>
> $$R' < R \qquad \text{and} \qquad \frac{1}{3} S.R' < \frac{1}{3} SR$$
>
> In other words $V_1 < V$.

We note that the nature of the polyhedron is not specified by al-Khāzin; but his demonstration supposes that this polyhedron is circumscribed on a sphere, which is the case for a regular polyhedron, but the demonstration is not valid for all polyhedra and solids. As we can see, al-Khāzin's approach is different in the case of space and in the case of planes: this time, he does not compare polyhedra of the same area but with a different number of faces. On the contrary, he arrives directly at a result using the formula which relates the volume of a sphere with its area, a formula that he obtains by approximating the sphere with non-regular polyhedra.

About half a century after al-Khāzin, Ibn al-Haytham, who is not satisfied with his predecessors' work (although he does not name them), takes this question and writes a treatise on isoperimetrics.[32] In the introduction he writes: 'The mathematicians mentioned and used this notion but any demonstration that could be due to them has not reached our times, or any convincing proof.' One is perplexed by such a statement—at least in the present state of our knowledge. Did Ibn al-Haytham not know of the treatise of al-Khāzin? Did he find it insufficient? Finally, who were these mathematicians? In any case Ibn al-Haytham intends to give 'a universal demonstration'.

An analysis of this text shows that, contrary to al-Khāzin, Ibn al-Haytham was looking for a *dynamic* approach which, successful in the case of isoperimetric problem, fails for isepiphanies because of the limited number of regular polyhedra. But this was a fertile failure: although it stopped him from achieving his aim in the case of isepiphanies, it allowed him to propose an original theory of the solid angle, the first to deserve this title.

The first part of this treatise, which was the vanguard of mathematical research in the period of Ibn al-Haytham and also for some centuries after him, is dedicated to plane figures. The author rapidly regulates this case. Like al-Khāzin, he starts by comparing regular polygons of the same perimeter and a different number of sides, and demonstrates the following.

1 Let P_1, P_2 be two regular polygons, with n_1, n_2; A_1, A_2; p_1, p_2, respectively the number of their sides, their areas and their perimeters.

If $p_1=p_2$ and $n_1<n_2$, then $A_1<A_2$

2 Let p be the perimeter of a circle, A its area, p' the perimeter of a regular polygon and A' its area.

If $p=p'$, then $A>A'$

Contrary to al-Khāzin and all his known predecessors, Ibn al-Haytham uses the first proposition here to establish the second, considering the circle as the limit of a series of regular polygons; i.e. he follows what we call a dynamic approach. In fact, from these two propositions, he shows that, of all plane figures with a given perimeter, the disc is the one with the largest area. During this demonstration, he assumes the existence of a limit—the area of the disc—which was established from *The Measurement of the Circle* of Archimedes.

The second part, dedicated to isepiphanies, opens with ten lemmas which constitute a treatise about the solid angle and whose analysis goes far beyond our limits here. These lemmas establish two propositions—5a and 5b in the first edition of this text[33]—which allowed him to reach his conclusion. Let us pause as briefly as possible on these two propositions.

5a Of two regular polyhedra having similar faces and equal surfaces, the one which has the greater number of faces will also have the larger volume.

Let A (respectively B) be the centre of a circumscribed sphere on the first (respectively second) polyhedron; AE (respectively BG) is the distance from the centre to the plane of one face, S_A (respectively S_B) is the total area of the polyhedron and V_A (respectively V_B) is its volume (Figure 13.9). We have

$$V_A=\frac{1}{3}S_A.AE \qquad \text{and} \qquad V_B=\frac{1}{3}S_B.BG$$

By hypothesis we have $S_A=S_B$. Let n_A and n_B be the number of faces of the polyhedra; if we have $n_B>n_A$, then $V_B>V_A$.

Ibn al-Haytham's demonstration compares AE and BG. To arrive there, he considers the bases of pyramids A and B which he decomposes into triangles. The reasoning is from results already established for solid angles having peaks at the centres of the spheres.

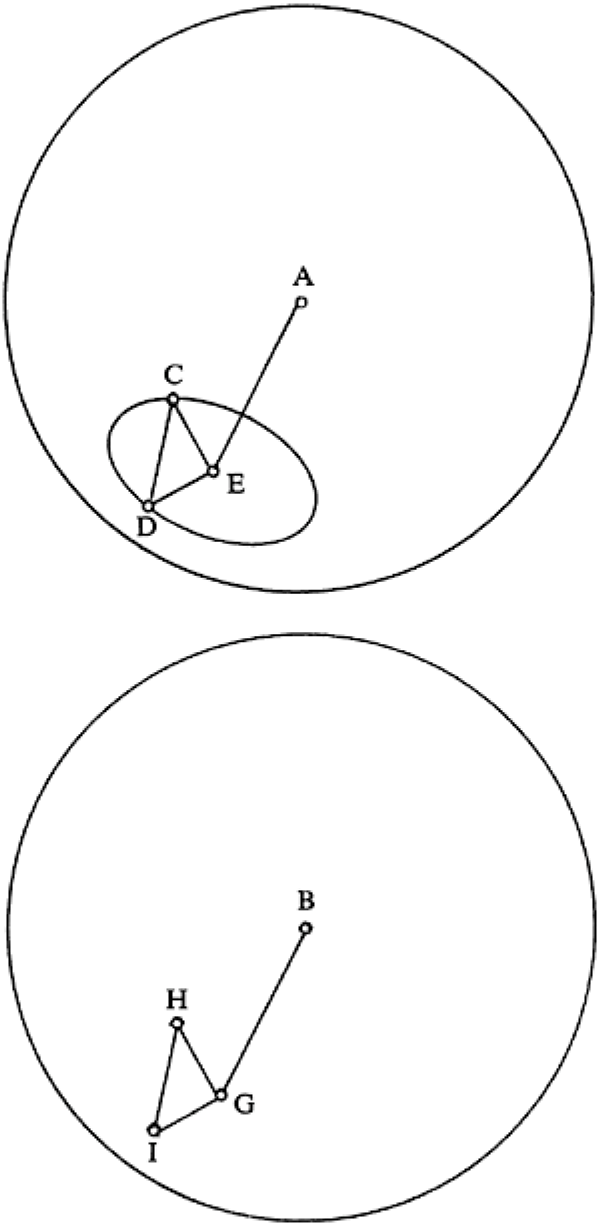


Figure 13.9

5b If two regular polyhedra have similar regular polygons for faces, and are inscribed in the same sphere, then the one that has the larger number of faces also has the larger surface and the larger volume.

For a better illustration of Ibn al-Haytham's approach let us follow the most prominent steps of his demonstration.

Let P_1 and P_2 be two polyhedra, S_1 and S_2 their surfaces, V_1 and V_2 their volumes, and n_1 and n_2 the numbers of their faces, assuming that $n_1 > n_2$

If A is the centre of the sphere circumscribed on the two polyhedra, we shall have n_1 equal pyramids, with peak A, associated with the faces of P_1, and n_2 regular pyramids associated with the faces of P_2.

Let α_1, s_1, h_1, respectively, be the angle at the peak, the area of the base, and the height of a regular pyramid P'_1 associated with P_1, and α_2, s_2, h_2 the elements of a regular pyramid P'_2 associated with P_2. We have

$n_1\alpha_1 = n_2\alpha_2 = 8$ right solid angles

But, since $n_1 > n_2$, we have $\alpha_1 < \alpha_2$.

We can assume that pyramid P'_1 and pyramid P'_2 have the same axis. Since $\alpha_1 < \alpha_2$, the solid angle of P'_1 is interior to the solid angle of P'_2, the edges of P'_1 cut the sphere behind the base plane of P'_2. The planes of the two bases are parallel and cut the sphere in circles circumscribed on these bases; we then deduce that

$s_1 < s_2$ and $h_1 > h_2$

However, we have

$$\frac{\alpha_1}{8D} = \frac{s_1}{S_1} = \frac{1}{n_1} \qquad \text{and} \qquad \frac{\alpha_2}{8D} = \frac{s_2}{S_2} = \frac{1}{n_2}$$

whence

$$\frac{\alpha_2}{\alpha_1} = \frac{s_2 S_1}{s_1 S_2}$$

Now, Ibn al-Haytham had established before in a lemma that $\alpha_2/\alpha_1 > s_2/s_1$. Consequently

$$\frac{s_2}{s_1} \cdot \frac{S_1}{S_2} > \frac{s_2}{s_1}$$

whence $S_1 > S_2$. But we know that

$$V_1 = \frac{1}{3} S_1 h_1 \qquad \text{and} \qquad V_2 = \frac{1}{3} S_2 h_2$$

Now $S_1 > S_2$ and $h_1 > h_2$; therefore $V_1 > V_2$.

We have just seen that Ibn al-Haytham starts from regular polyhedra. The two propositions 5a and 5b only apply to the case of the tetrahedron, the octahedron and the icosahedron, as the number of faces of a regular polyhedron with square or pentagonal faces is fixed (six or twelve). Proposition 5a therefore means that, if a regular tetrahedron, octahedron and icosahedron have the same areas, then their volumes increase in the following order: tetrahedron, octahedron and icosahedron. Proposition 5b means that if a regular tetrahedron, octahedron and icosahedron are inscribed in the same sphere, their volumes increase in this same order.

Ibn al-Haytham's intention can be clearly understood from the above: from a comparison between polyhedra with the same areas and different numbers of faces, to establish the 'extreme' property of the sphere; i.e. to approximate the sphere as the limit of inscribed polyhedra.

But this dynamic approach goes against the finitude of the number of regular polyhedra, and this fact, we acknowledge, remains incomprehensible. It seems in fact that Ibn al-Haytham did not see that these polyhedra reduce to those of Euclid, and thus their number is finite—an oversight that to us is inexplicable since very few mathematicians knew the *Elements* of Euclid as profoundly as Ibn al-Haytham.[34] But, as we have seen, this failure comes with a great success: his theory of the solid angle.

In the present state of our knowledge, these two contributions—from al-Khāzin and from Ibn al-Haytham—are by far the most important in Arabic mathematics, at a level which is far from being reached by their successors such as Ibn Hūd, Jābir ibn Aflaḥ, Abū al-Qāsim al-Sumaysāṭī, among others. Although the last mentioned considers the isoperimetric problem,[35] Ibn Aflaḥ only considers the isepiphanies and only includes regular polyhedra in his demonstration.[36] Future research will tell us whether there were other contributions in the class of al-Khāzin and Ibn al-Haytham and whether elements of this subject were transmitted to Latin mathematics.[37]

NOTES

1 Cf. Rashed (1987).
2 Cf. Thābit ibn Qurra, *Œuvres d'Astronomie,* pp. 68–82.
3 Rashed (1993c), pp. 495–510.
4 Euclid, *The Elements,* pp. 258–9.
5 See the article 'Banū Mūsā', *Dictionary of Scientific Biography,* 1970, vol. 1, pp. 443–6, and the treatise of the Banū Mūsā edited, translated and commented in Rashed (1995), Chapter I.
6 Cf. Rashed (1993d, b).
7 Cf. Rashed (1993b) and (1995b) where we find a complete analysis of the Banū Mūsā's text.

8 MS Le Caire, no. 40 Riyāḍa, fol. 180[v].
9 Youschkevitch (1964a).
10 Rashed (1993a).
11 Rashed (1993c).
12 Rashed (1993c), and Rashed (1981).
13 Heath, vol. 1, pp. 183–200. The text of Simplicius was translated into German by Becker (1964:29 *et seq*.). See also Becker (1936).
14 It concerns the treatise of Ibn al-Haytham *On the Lunules (Fï al-hilāliyyāt)*. This text has been edited, translated into French and commentated in Rashed (1993c), pp. 69–81; commentary on pp. 32–4. Later, in a second treatise, Ibn al-Haytham mentions his first text: 'I wrote a short treatise about the lunules with the help of particular methods' (Rashed 1993c:102).
15 Ibn al-Haytham writes in his first treatise about 'the Ancients'; but he does not reproduce in a strict sense any figures of Hippocrates. His first result remains a slight generalization of one of the propositions of Hippocrates quoted by Simplicius from a text of Alexander, which complicates the problem remarkably. It is about proposition 3 of the first treatise—see below—which also occurs in his text on the quadrature of the circle and in his second treatise, proposition 8.
16 Cf. Suter (1899).
17 Cf. Rashed (1991b).
18 This treatise entitled *On the Figures of Lunules,* has been edited and translated in Rashed (1993c), pp 102–75; commentary on pp. 37–68.
19 Cf. Rashed (1991c).
20 It concerns the evidence of Simplicius, cf. Simplicius, VII, 4/2, lines 12–17: 'It was shown, not only before Aristotle, as he uses it as established [proposition], but also by Archimedes, and in a more detailed way—πλατύτερου—by Zenodorus that, among the isoperimetric figures, the largest one of the planar ones, is the circle and of the solids, the sphere.'

This text, as Schmidt (1901:5–8) has noted, shows that the fundamental propositions were known before Zenodorus—it was Schmidt who called the attention of the historians of science to the text of Simplicius. This idea led Mogenet to assert that Zenodorus only stated 'in a general way'—πλατύτρων—the doctrine of isoperimeters and to argue that the mathematician should be placed in the third century before our era. Cf. Mogenet.
21 About the dates of Zenodorus, we are not much more advanced today than yesterday: after Archimedes and before Pappus. In his *Commentaires de Pappus et Théon d'Alexandrie sur l'Almageste,* pp. 354 *et seq*., Rome takes account of this uncertainty and places him between the second century before our time and the third century after our time. We shall not take up this controversy here, which involves Cantor, Schmidt, Mogenet, among others are connected. Recently by a faulty compilation of Diocles, which survived in Arabic, one could have thought that this could have brought a new element to this problem; but that was not the case.

For the text of Zenodorus, cf. Rome and also the translation of *Commentaire de Théon d'Alexandrie sur le premier livre de la composition mathématique de Ptolémée,* by Halma.
22 Cf. Schmidt (1901).

23 Cf. Ptolemy, pp. 9–10.
24 Cf. Pappus, Book 5, pp. 239 *et seq*.
25 Ptolemy, p. 10. Note that, in the Arabic translation of al-Ḥajjāj, at the beginning of the ninth century, MS Leiden 680, we read (folios 3^v–4^r): ‘Since amongst the polygonal figures which are inscribed in equal circles those which have the largest number of angles are the largest, the circle is the largest of the planar figures and the sphere is the largest of the solid figures.’
26 Theon of Alexandria, p. 33.
27 MS Istanbul, Aya Sofia 4830, 53^r–80^v, fol. 59^v. Cf. al-Kindī, *Fī al-Ṣināʿat*, p. 41.
28 As al-Kindī wrote: *‘Kamā ʾawḍhaḥnā fī kitābinā fī al-ʿukar’*.
29 Cf. al-Nadīm, p. 316.
30 See the edition and the translation of Lorch (1986:150–229).
31 It is surprising that neither Theon, nor Pappus, nor the following historians noticed this error, which was only revealed by Coolidge (1940), p. 49. Let us consider this lemma in a different language. It searches for the maximum of

$$ax + by \qquad \text{when} \qquad \sqrt{(a^2 + x^2)} + \sqrt{(b^2 + y^2)} = 1$$

It is necessary that $ax'+by'=0$, whence $x'=b$ and $y'=-a;$ and from the second equality

$$\frac{bx}{\sqrt{(a^2 + x^2)}} = \frac{ay}{\sqrt{(b^2 + y^2)}}$$

Putting $x=au$ and $y=bv$, we have

$$\frac{u}{a\sqrt{(1 + u^2)}} = \frac{v}{b\sqrt{(1 + v^2)}}$$

whilst the statement has $u=v$.
32 Rashed (1993c:331–439). We find there a critical edition of text of Ibn al-Haytham, its French translation as well as an analysis.
33 *Ibid.*
34 To be convinced it is enough to read his *Commentary on Euclid Postulates* and his *Doubts concerning the Book of Euclid*.
35 The text of Abū al-Qāsim al-Sumaysāṭī can be found in many manuscripts. See our edition, French translation and analysis in Rashed (1995), Chapter IV. They are often collections which include ‘the intermediate books’ *al-Mutawassiṭāt*, aimed at an educated public and for the teaching of astronomy.
36 Jābir ibn Aflaḥ: *Iṣlāḥ al-Majisṭī*, MS Escorial 390, 12^r–12^v.

37 Everyone knows that the book of Jābir ibn Aflaḥ was translated into Latin. Other facts should be examined, as for example a proposition which is found in *Geometria speculativa,* Book II, of Bradwardine, which we find next in *De subtilitate* of Cardano, and which is no other than proposition 6 of al-Khāzin: ‘of all the isoperimetric plane figures with the same number of sides and equal angles, the largest has its sides equal’. Is it a common source, an independent deduction or a transmission?

14
Geometry

BORIS A.ROSENFELD AND ADOLF P.YOUSCHKEVITCH

INTRODUCTION

The first written evidence concerning geometry in Arabic dates back to the end of the eighth and the beginning of the ninth century. Geometry was written in Arabic, which was generally used by scholars of the Islamic countries, as well as in Syriac and Persian, languages which were used much less in scientific practice, and it convincingly testifies that the ancient Greek and Hellenistic tradition along with the Indian tradition, which in part also followed the Greeks, exerted a dominant influence on geometry, on other branches of mathematics and on exact sciences in general.

However considerable was this influence, the Arabic geometry, even at the early stage of its development, acquired its own specific features. This happened with respect to the place of geometry in the system of mathematical sciences, to its interconnections with the other branches of mathematics, especially with algebra, and to the interpretation of both known and quite new problems. Combining various elements of the early or late Greek heritage and absorbing the knowledge of other nations, the Arab scholars outlined new directions for geometric ideas and enriched the adopted conceptions by their own thoughts, thus creating a new type of geometry and of mathematics in general.

From the ninth century onwards, many isolated contributions belonging to the Arabic literature were devoted to geometry. This discipline was also treated in works chiefly studying other mathematical sciences. All, or almost all, relevant literature falls under the following main categories.

1 Theoretical writings in geometry, both original and translated from other languages, dealing with the entire field of the science or discussing its separate sections.

   These include, first and foremost, Euclid's *Elements,* which gave birth to a great number of commentaries, many of them quite original and forming independent fields of research. A qualification remark is needed, however. The *Elements* are known to consist of thirteen books, many of which in spite of using geometric terminology are not essentially geometric; Book V is devoted to the general theory of ratios and proportions, Books VII–IX are arithmetical and number-theoretic and, finally, Book X contains a theory of some types of quadratic irrationalities. The other books of the *Elements* are geometric: Books I–IV and VI treat plane geometry and stereometry is studied in Books XI–XIII.

   Some tracts written by Archimedes also pertain to geometry. Most of these are considered in the chapter on the application of infinitesimal methods to quadratures and cubatures. Finally, we ought to mention Apollonius's *Conics* and Theodosius's *Sphaerica* as well as Menelaus's work of the same title.

   The influence of all the mentioned works as well as of other Greek writings, a good part of whose Arabic translations are lost, was indeed great.

2 Geometric contributions mainly devoted to other sciences—e.g. algebra, astronomy,

statics and optics—and philosophical treatises or general encyclopedic works. In this category falls Ptolemy's *Almagest,* where geometry is treated in the second part of Book 1; numerous Arabic *zijes* (astronomical tables), usually including theoretical sections complete with geometric rules; and treatises on astronomical instruments.

3 Treatises on practical geometry for land surveyors, builders, artisans etc. containing rules for geometric calculations and constructions and illustrating them by examples. These sources did not provide the appropriate proofs.

We do not claim that our subdivision of the geometric literature is comprehensive but we believe that it will be useful for general orientation.

GEOMETRY AND ALGEBRA

We shall first deal with the earliest known Arabic geometric work, or, rather, with a large geometric section of Muḥammad ibn Mūsā al-Khwārizmī's (*c.* 780–*c.* 850) algebraic treatise discussed in the chapter on algebra.

Of special interest is the 'Chapter on measurement' (*Bāb al-misāḥa*) of al-Khwārizmī's algebraic treatise. It is the earliest known Arabic text where algebra is applied in order to solve geometric problems, e.g. to calculate by the Pythagorean proposition the height of a triangle given its sides. Hero of Alexandria, in his *Metrica,* solved similar problems, but in a different way. All this, along with other rules and with the method of solving quadratic equations convincingly, testifies that the Arabic geometry adopted Hellenistic and, therefore, ancient Greek ideas. In particular, al-Khwārizmī's tests for distinguishing between acute, obtuse and right triangles coincide with those due to Hero which, in turn, can be traced back to Euclid's *Elements*. The same is true of the classification of quadrangles.

By indicating that the area of a regular polygon with any number of sides is equal to the product of its semi-perimeter and the semi-diameter of the inscribed circle, al-Khwārizmī explains that the area of a circle is equal to the product of its semi-diameter and the semi-circumference. For the ratio π of the circumference to its diameter al-Khwārizmī gives the values $\pi = 3\frac{1}{7}$, $\sqrt{10}$ and 62,832/20,000.

Archimedes introduced the first value in his *Measurement of the Circle,* the Chinese astronomer Chang Hêng (78–139) and, later on, the Indian astronomer Brahmagupta (b. 598) suggested the second one while the third value is due to another Indian astronomer, Āryabhaṭa (b. 476).

Al-Khwārizmī approximated the area of a circle as

$$S = d^2 - \frac{1}{7} d^2 - \frac{1}{2}\frac{1}{7} d^2$$

where *d* is the diameter. His rule corresponded to the value of $\pi = 3\frac{1}{7}$ which was also known to Hero. In addition al-Khwārizmī introduced the exact rule for the area a of a circle's segment with base *l,* height *h* and arc *s:*

$$\sigma = \frac{d}{2}\frac{s}{2} - \left(\frac{d}{2} - h\right)\frac{l}{2}$$

The first term of this expression represents the area of the corresponding sector and the second term represents the area of the triangle that makes up the difference between the sector and the segment. Al-Khwārizmī also offered rules for calculating the volumes of a prism, a pyramid, a cylinder and a cone. He also considered the truncated pyramid, giving its volume as the difference between the volumes of two appropriate complete pyramids, but he did not calculate the volume of a sphere.

Many Arabic manuals on arithmetic and algebra included sections similar to al-Khwārizmī's 'Chapter on measurement'. Abū al-Wafā' (940–998) included a rather large number of geometric rules in his arithmetical treatise *A Book about What is Necessary for Scribes, Employees and Others from the Science of Arithmetic* (*Kitāb fī mā yaḥtāju ilayhi al-kuttāb wa-l-'ummāl wa ghayruhum min 'ilm al-ḥisāb*). As compared with al-Khwārizmī the author added new information, doubtlessly borrowed in part from Greek and Indian sources (the Archimedes-Hero's rule for calculating the area of a triangle given its sides, the Indian rule for approximately calculating the side of a regular inscribed polygon by the number of the sides and the diameter of the circumscribed circle). This part of Abū al-Wafā's book directly borders on the geometric section of *The Sufficient in the Science of Arithmetic* (*al-Kāfī fī 'ilm al-ḥisāb*) *by* al-Karajī (d. *c.* 1030).

Thus, while applying elementary geometric constructions in order to solve numerically quadratic equations and introducing algebraic methods so as to calculate geometric quantities, Arab mathematicians created a kind of bridge connecting algebra with geometry. Obviously, they did not represent the real roots of arbitrary algebraic equations by the co-ordinate lines of the points of intersection of appropriately chosen algebraic curves. This was done later at the end of the seventeenth century. Nonetheless, in the particular case of cubic equations, Arab mathematicians, 'Umar al-Khayyām and Sharaf al-Dīn al-Ṭūsī especially (see the chapter on algebra), anticipated this idea. Jamshīd al-Kāshī (d. *c.* 1430), in his *Key of Arithmetic* (*Miftāḥ al- ḥisāb*), claimed that he introduced such a connection for all equations of the fourth degree (with positive roots). But even supposing that this treatise were actually written, it has not yet been found.

GEOMETRIC CALCULATIONS

From dealing with the interrelation between geometry and algebra and touching on the problems of measuring some figures it is natural to turn to other geometric calculations. We shall not dwell on the infinitesimal calculations of quadratures and cubatures as made by Thābit ibn Qurra, his grandson Ibrāhīm ibn Sinān and Ibn al-Haytham, since these are discussed in the chapter on infinitesimal methods. Instead, we shall continue reviewing al-Khwārizmī's exact and approximate calculations.

In this field the Arabs were quick to learn the Greek's heritage and, moreover, to considerably enrich it, witness *The Book of the Knowledge of Measuring Plane and Spherical Figures* (*Kitāb ma'rifa misāḥat al-ashkāl al-basīṭa wa-l-kuriyya*) written in the mid-ninth century by brothers Muḥammad (d. 872), Aḥmad and al-Ḥasan Banū Mūsā. The brothers gave rules for calculating the areas of regular polygons circumscribed around and inscribed into a circle and the area of a circle calculated as 'a plane figure [i.e. a product] of the semidiameter of the circle and a half of its circumference'. The authors proved that for all circles 'the ratio of the diameter of a circle to its circumference is the same' and that the ratio of a circumference to the diameter of its circle is more than the ratio of $3\frac{10}{71}$ and less than the ratio of $3\frac{1}{7}$. Archimedes, in his *Measurement of the Circle,* was the first to prove both these inequalities.

The authors went on to demonstrate the Archimedes-Hero theorem on deriving the area of a triangle by its sides. In their following propositions they ascertained that: the lateral surface of a circular cone is 'a plane figure' (a product) of its generator and a half of the circumference of its base; a section of a circular cone by a plane parallel to the base is a circle; the lateral surface of a truncated circular cone is 'a plane figure' of its generator and the half-sum of the circumferences of its bases; the surface of a hemisphere is equal to twice the area of the sphere's great circle; the volume of a sphere is the product of its semi-diameter by one-third of its surface. The authors proved the two last theorems by *reductio ad absurdum*. All their propositions are due to Archimedes who proved them in his work *On the Sphere and the Cylinder*.

Finally, the Banū Mūsā considered three theorems discussing two classical Greek problems of deriving two mean proportionals, x and y, between two given quantities, a and b $(a/x=x/y=y/b)$, and the trisection of an angle, and described the extraction of cubic roots in sexagesimal numbers. They offered two solutions of the first problem. One of these, due to Archytas, by considering the intersection of three solids of revolution, a cylinder, a cone and a torus, actually provided the proof that a (stereometric) solution does exist. With regard to the trisection, it follows Archimedes's method introduced in his *Lemmas*. For a further discussion of this problem, see Rashed (1995: chap. I).

Thābit ibn Qurra, a student of brothers Banū Mūsā, was the author of the above-mentioned treatises on the quadratures and cubatures made by means of infinitesimal methods and a *Book on the Sections of a Cylinder and on its Surface* which was based on these methods. In addition, he wrote two treatises on geometric calculations, a partly extant *Book on the Measurement of What is Cut off by Lines* (*Kitāb fī misāḥa qaṭ' al-khuṭūṭ*) and a completely preserved *Book on the Measurement of Plane and Solid Figures* (*Kitāb fī misāḥat al-ashkāl al-musaṭṭaḥa wa-l-mujassama)*. In the first writing Ibn Qurra measured the part of a circle delimited by the sides of an equilateral triangle and a regular hexagon, both of them inscribed into the circle. The author considered three cases (Figures 14.1(a), (b), (c)

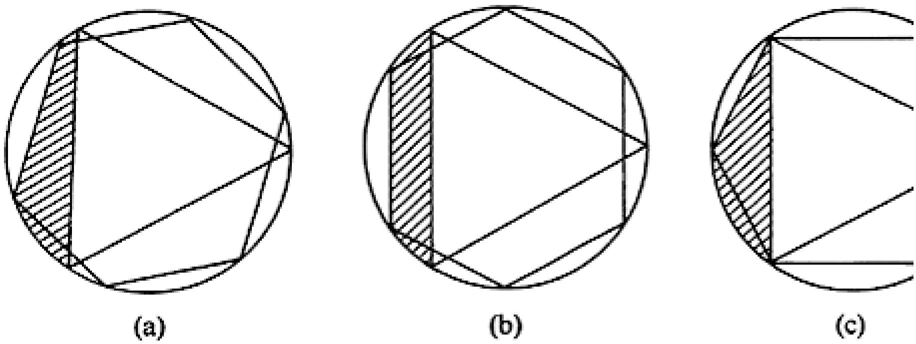


Figure 14.1

respectively) and proved that the area of the figure in question was equal to $\frac{1}{6}$ of the area of the circle.

The second book included many rules for calculating areas and volumes and, in particular, volumes of 'solids with differing bases', i.e. of truncated pyramids and cones. Let the area of their bases be S_1 and S_2 and their height be h. Then, in either case, the volumes of these solids are given by

$$V = \tfrac{1}{3}h\,[S_1 + \sqrt{(S_1 S_2)} + S_2].$$

Thābit ibn Qurra proved this rule in his *Book on the Measurement of Parabolic Solids (Maqāla fī misāḥat al-mujassamāt al-mukāfiya).*

It is impossible and unnecessary to describe the calculations of the elements of various figures and solids, especially regular polygons and polyhedra made ever more accurately by numerous Arab scholars. If the sides of polygons were quadratic irrationalities the scholars derived them by solving quadratic equations and extracting roots, sometimes repeating these procedures a few times over. The same method was used to determine the edges of regular polyhedra since all of them were, as Euclid showed in Book XIII of his *Elements,* quadratic irrationalities.

When the sides of the polygons were cubic irrationalities they were evaluated by solving cubic equations, which was done by considering the intersection of conic sections and equivalent methods or, alternatively, by approximate calculations. This case included finding the sides of regular polygons with seven, nine and 180 sides. The last polygon was important for compiling trigonometrical tables since half its side was Sin 1°=R sin 1°.

As noted in the chapter on combinatorial analysis, Arab mathematicians attained extremely high perfection in calculations. This is especially true with regard to Ulugh Beg's Samarkand school. Most prominent in calculations were two of al-Kāshī's works. In Book IV of his *Key of Arithmetic* he compiled a large number of rules for determining areas of plane figures such as triangles, quadrangles and regular polygons, of the circle and its sector and segment and of other more complicated figures, as well as volumes and lateral surfaces of truncated pyramids and cones, of the sphere and its segment, of regular polyhedra etc. Al-Kāshī used the approximate value of π represented by the sexagesimal fraction $3°8'29''44'''$=3.141593. He then described the measurement of the volumes of bodies given their weights and adduced an extensive table of the specific gravity of various substances. Al-Kāshī paid special attention to the method of measuring the parts of edifices and buildings such as arches, vaults, hollow cupolas and stalactite surfaces widely wonted in the medieval East. While measuring the volumes of truncated cones and hollow cupolas he applied methods of approximate integration, as we would say in our time.

The other writing of al-Kāshī, his *Treatise on the Circumference (al-Risāla al-muḥīṭiyya)*, was the summit of calculating mastery. Here, he determined the value of π with an accuracy that surpassed by far not only all previous attempts, but even the achievements of many later European scholars (see later). Al-Kāshī calculated π in the same way as Archimedes did in his *Measurement of the Circle,* which was translated into Arabic even in the ninth century (the brothers Banū Mūsā described Archimedes's calculations).

Striving to achieve a very high precision of his calculations, al-Kāshī considered inscribed and circumscribed regular polygons with 3×2^{28}= 805,306,368 sides whereas Archimedes restricted himself to polygons with $96=3\times2^5$ sides.

Denote the side of a polygon with n apexes by a_n and let b_n be such a chord of the appropriate circumscribed circle that (Figure 14.2)

$$a_n^2+b_n^2=(2R)^2.$$

Then

$$a_{2n}=\sqrt{\{2R^2-R\sqrt{[(2R)^2-a_n^2]}\}}.$$

It was these b_n's rather than the a_n's themselves that al-Kāshī actually determined. Using the rule

$$b_{2n}=\sqrt{[R(2R+b_n)]}$$

he reduced the calculation of a_k, $k=3\times2^{28}$, to extracting a square root twenty-eight times in succession. Al-Kāshī selected this value of k since the difference between the perimeters of the polygons, inscribed into and circumscribed around a circle whose diameter D was 600,000 times the diameter of the Earth, would then be less than the breadth of a horse hair. Now, in al-Kāshī's opinion, D was the diameter of the sphere of the fixed stars, so that natural scientists would never encounter a larger circle. He accomplished his calculations in sexagesimal fractions whose use made the extraction of roots easier as compared with the use of decimal fractions.

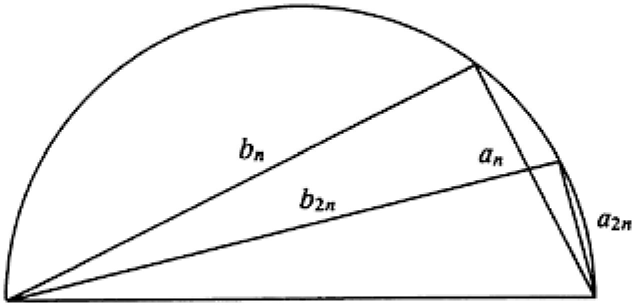


Figure 14.2

After determining the perimeter of the inscribed polygon with 3×2^{28} sides, al-Kāshī calculated it for the appropriate circumscribed polygon and assumed that the length of the circumference was equal to the arithmetical mean of these perimeters. He got

π=3;8, 29, 44, 0, 47, 25, 53, 7, 25

and converted this value into the decimal system arriving at

π=3.14 159 265 358 979 325.

Of the seventeen decimal places only the last one is wrong (the correct value is 38). In Europe, only 150 years later, the Dutch scientist A. van Roomen attained the same precision in determining π. He considered inscribed and circumscribed polygons with 2^{30} sides.

Note that al-Kāshī also determined the sine of 1° with the same degree of accuracy as π. He represented it as a root of a cubic equation and solved this equation by means of his own rapidly converging iterative method.

Incidentally, Arabic mathematicians repeatedly expressed their belief that the ratio of the length of a circumference to its diameter was irrational.

However, even Abū al-Rayḥān al-Bīrūnī (973–1048), in his *Canon Masudicus* (*al-Qānūn al-Mas'ūdī*), claimed that the ratio of 'the number of the circumference' to 'the number of the diameter [which he took to be equal to 2] is irrational' (pp. 126, 510).

Subsequent European mathematicians were also sure that π was irrational but only J.H.Lambert, a native of Alsace, in 1766 succeeded in proving this. The symbol π was proposed by W.Jones and it was introduced into general use by L.Euler. In 1882 the German mathematician F.Lindemann specified the arithmetical nature of the number π by proving that it was transcendental.

It is a typical feature of the Arabic geometry that it restored to life on a higher scientific level the tendencies of masterly calculations that went back to ancient Babylon and were later developed in the Hellenistic countries, after Archimedes' work.

GEOMETRIC CONSTRUCTIONS

Each society had to occupy itself with geometric constructions necessary for land surveying and construction as well as with geometric calculations. In these constructions, the stretched string played the same role as the one occupied in our time by the ruler and compasses. In particular, right triangles with legs three and four parts long (and a hypotenuse five parts long) were constructed by means of a string divided into twelve equal parts. According to legend, the Egyptian *harpedonapts,* or stretchers of the string, instructed Democritus in geometry. And, as witnessed by the ancient Indian *Śulbasūtras,* such strings were used in order to construct altars.

The Greeks ascribed the invention of the compasses to Thales. Euclid, in his *Elements,* invariably accomplished his constructions by ruler and compasses, and he considered only such intervals as could be constructed from integral segments by means of these instruments. All irrationalities that occur in this classical work are therefore quadratic.

In the fourth century BC the Greeks started to use instruments for constructing cubic irrationalities, especially the *neusis,* a ruler with two marked points. It was by means of such a ruler that Archimedes, in his *Lemmas,* trisected angles by reducing this problem to solving cubic equations.

The Greeks used special curves in order to solve some antique problems geometrically, i.e. to construct appropriate segments or angles. For example, in the fourth century BC Menaechmus applied conic sections to duplicate cubes. The same sections were also applied so as to solve a more general problem of constructing two mean proportionals between two given segments. In the second century BC Nicomedes and Diocles introduced the conchoid and the cissoid for the same purposes. The former was used to trisect angles and the latter to duplicate cubes. According to our terminology, both curves are of the third degree. Hippias of Elis, back in the fifth century BC, trisected angles by means of the quadratrix, a transcendental curve. In the next century, Dinostratus used this curve to construct a segment reciprocal to π so as to square the circle, i.e. to construct a square equivalent to a given circle. All these curves, as well as the spiral of Archimedes, also used for squaring the circle, were studied in many theoretical investigations, especially in modern Europe.

In the Arabic manuscripts known to us there are many instances when conic sections were used to construct segments and angles. However, there is no evidence of any curves other than those mentioned above. And still, the Spanish Jew Alfonso, in his treatise *Straightening the Curved* (*Meyashshēr ʿāqōb*), written in the fourteenth century under a strong influence of Arab mathematicians used the conchoid to trisect angles and construct two mean proportionals (*Meyashshēr ʿāqōb*, pp. 82–4).

Thābit ibn Qurra devoted two writings to geometric constructions. In the *Treatise on the Proof, Ascribed to Socrates, Concerning the Square and its Diagonal* (*Risāla fī al-ḥujja al-mansūba ilā Suqrāṭ fī al-murabbaʿ wa quṭrihi*) he solves the problem of how to cut a square constructed on the hypotenuse of a right triangle in such a way as to make up squares constructed on the legs of the same triangle. Figure 14.3 reproduces one of Thābit ibn Qurra's drawings. Here, the square BCHJ is constructed on the hypotenuse of triangle ABC; it is then cut into parts from which the figure

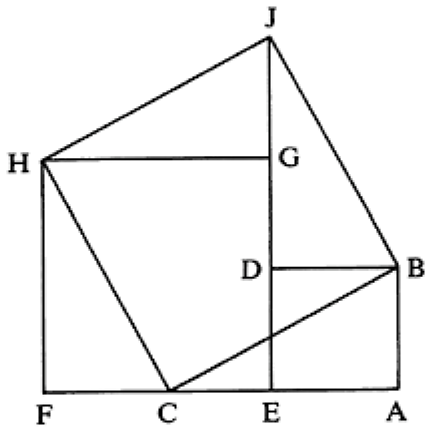


Figure 14.3

BAFHGD is made up. This figure indeed consists of squares ABDE and EFHG constructed on the legs of triangle ABC.

In his *Book on the Construction of a Solid Figure with Fourteen Faces Inscribed into a Given Sphere (Kitāb fī al-ʿamal shakl mujassam dhī arbaʿ ʿashara qāʿida tuḥīt bihi kura maʿlūma),* the same author considers a spatial construction of a polyhedron bounded by six squares and eight equilateral triangles. This solid could be obtained from a cube if all of its apexes are truncated by cutting off a half of each edge of the cube adjacent to each apex.

Being bounded by regular polygons of two types, this solid is one of the thirteen so-called 'semi-regular' polyhedra. Alī of these were discovered by Archimedes.

Two books were expressly devoted to geometric constructions: *The Book of Spiritual Clever Tricks and Mysteries of Nature on the Subtlety of Geometric Figures (Kitāb al-ḥiyal al-rūḥāniyya wa-l-asrār al-ṭabīʿiyya fī daqāʾiq al-ashkāl al-handasiyya)* by the noted philosopher Abū Naṣr al-Fārābī (*c.* 875–950) and *The Book about What is Needed for the Artisan from Geometric Constructions (Kitāb fī mā yaḥtāju al-ṣāniʿ min al-aʿmāl al-handasiyya)* by Abū al-Wafāʾ. The second writing almost completely incorporated the first one. Incidentally, the term *ḥiyal* meaning 'clever (or involved) tricks' also stood for mechanics, and especially for the doctrine of mechanisms and automatic devices. While discussing arithmetic, al-Fārābī used this term to denote algebra and, in dealing with such tricks in geometry, he meant the art of geometric constructions.

Taken together, both books included the following: (1) elementary constructions by ruler and compasses; (2) constructions by special instruments of two mean proportionals and the trisection of angles, these procedures are equivalent to solving some cubic equations; (3) construction by ruler and compasses of equilateral triangles, squares and regular polygons with five, six, seven, eight and ten sides (the construction of a polygon with seven sides is equivalent to solving a certain cubic equation and was made approximately; a regular polygon with nine sides was constructed by trisecting an angle);

(4) a number of constructions by ruler and compasses with a fixed spread; (5) a construction of a parabola (of 'a burning mirror') by graphically determining a number of its points; (6) transformations of one polygon into another; (7) spatial constructions; and (8) constructions on spheres and, in particular, constructions of the apexes of regular and semi-regular polyhedra.

Traditions going back to the ancient Indian *Śulbasūtras* doubtlessly influenced these two treatises and it also seems that *The Philosopher of the Arabs* (*Faylasūf al-'arab*) Ya'qūb al-Kindī (d. 873) was an intermediary link between these traditions on the one hand and al-Fārābī and Abū al-Wafā' on the other. Al-Kindī's writings are lost, but the Arab historian al-Qifṭī (1173–1248) described his *Book on Constructing the Figure of Two Mean Proportionals* (*Kitāb fī 'amal shakl al-muwassiṭayn*), *Book on Dividing Triangle and Quadrangle (Kitāb taqsīm al-muthallath wa-l-murabba'* and *Book on Dividing Circle into Three Parts (Kitāb qisma al-dā'ira bi-thalātha aqsām)* (al-Qifṭī, *Ta'rīkh al-ḥukamā'*, p. 371).

Other extremely interesting constructions were the transformations of a square into a sum of several squares and vice versa. The *Śulbasūtras* also contained problems of this kind solved by the Pythagorean theorem. Describing several versions of constructing a square equal to the sum of three other squares congruent to each other, al-Fārābī and Abū al-Wafā' criticize the inaccurate methods used for this purpose by artisans. One

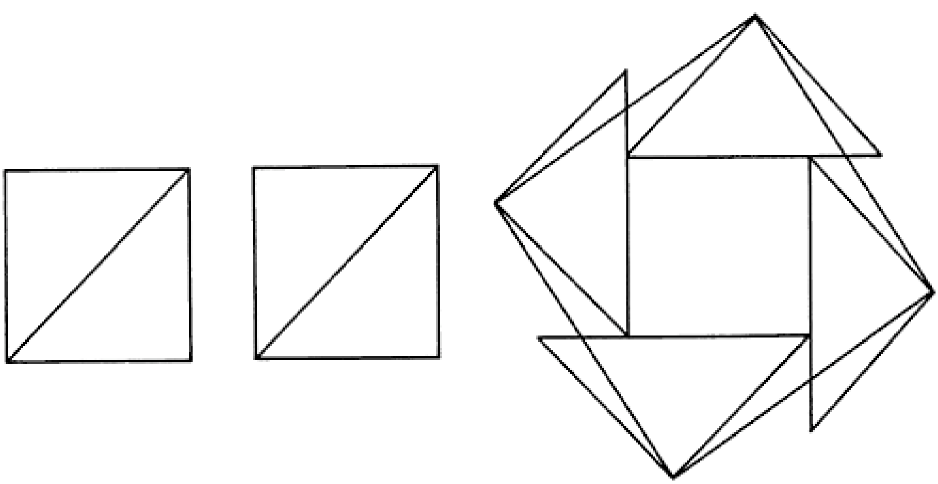


Figure 14.4

of the authors' tricks in solving this problem consisted in cutting each of the two given squares along its diagonal and placing the four triangles thus obtained adjacent to the third given square as shown on Figure 14.4. After this, the apexes of triangles lying opposite to the sides of this square were connected by straight lines, the parts of the triangles situated beyond these constructed lines were cut off and used to complement the figure to the constructed square.

Also of note was a spatial construction in which the side of a constructed square was equal to the diagonal of a cube whose edge was the side of the given square.

Ibrāhīm ibn Sinān ibn Thābit (908–946), a grandson of Thābit ibn Qurra, in his *Book on Constructing the Three Sections (Maqāla fī rasm al-quṭū' al-thalātha),* constructed parabolas (as al-Fārābī and Abū al-Wafā' did), ellipses and hyperbolas by graphically determining a number of their points. Other authors offered continuous constructions of conic sections. Thus, al- Ḥasan, one of the Banū Mūsā brothers, in his *Book on an Oblong Round Figure (Kitāb al-shakl al-mudawwar al-mustaṭīl*), constructed ellipses just as gardeners of our time are apt to lay out elliptic flower-beds. A string is fastened to two pegs and stretched tight by a third peg (Figure 14.5). This procedure is based on the (modern) definition of the ellipse according to which the sum of the two focal radius vectors of any point belonging to a given ellipse is constant.

Wayjan al-Kūhī (tenth-eleventh centuries) even devised a special instrument for continuously constructing conic sections. The perfect compasses *(al-birkār al-tāmm),* as he called it, had one arm of variable length while the other arm was fixed at a constant angle with respect to the plane of the drawing (Figure 14.6). When this instrument is rotated, its first arm circumscribes a conic surface whose intersection with this plane is a conic section. Denote the constant angle by α and the angle between the arms of the compasses by β. Then the constructed conic section has eccentricity

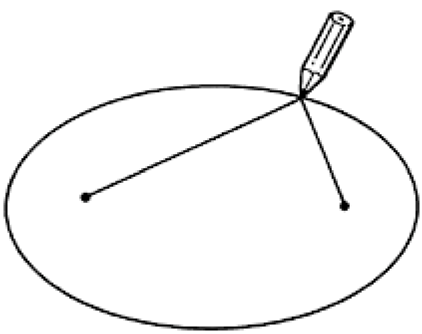


Figure 14.5

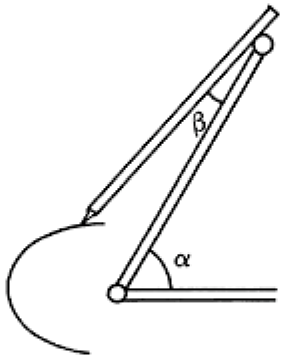


Figure 14.6

$\varepsilon=\cos\alpha/\cos\beta$. The section is an ellipse if $\alpha>\beta$, a parabola if $\alpha=\beta$ and a hyperbola if $\alpha<\beta$. Al-Kūhī described this instrument in his treatise *On* the *Perfect Compasses and on the Operations made by its Means (Fī al-birkār al-tāmm wa al-ʿamal bihi).*

As has been recently shown, Ibn Sahl, a mathematician from Baghdad, constructed a mechanical device for continously drawing conic sections (Rashed 1990). The Moroccan al-Ḥasan al-Marrākushī (d. 1262) who lived in Cairo expressly devoted a part of his encyclopedic *Book of Collected Principles and Results* (*Kitāb jāmiʿ al-mabādī wa-l-ghāyāt*) on constructing and applying astronomical instruments to geometric constructions and he described a large number of them.

From among numerous works on the construction of regular polygons with seven sides we shall mention al-Kūhī's *Treatise on Constructing the Side of an Equilateral Heptagon Inscribed into a Circle* (*Risāla fī ʿamal ḍilʿ al-musabbaʿ al-mutasāwī al-aḍlāʿ fī al-dāʾira*) and Abū ʿAlī ibn al-Haytham's (965-*c*. 1040) *Book on Constructing a Heptagon Inscribed into a Circle* (*Maqāla fī al-musabbaʿ fī al-dāʾira*). A regular polygon with nine sides was usually constructed by trisecting an angle of 60°. We also note al-Kūhī's*Treatise on Constructing an Equilateral Pentagon Inscribed into a Square (Risāla fī ʿamal mukhammas mutasāwī al-aḍlāʿ fī murabbaʿ maʿlūm).* Here, the author constructs an equilateral, though not a regular pentagon. It is inscribed into a square in such a way that its uppermost apex coincides with the mid-point of the upper side of the square; two sides of the pentagon meeting in this apex end on the lateral sides of the square; and the two other apexes are placed on the lower side of the square (Figure 14.7). This problem which might be reduced to an equation of the fourth degree is solved by considering the intersection of two hyperbolas.

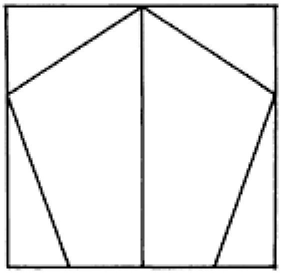


Figure 14.7

THE FOUNDATIONS OF GEOMETRY

Euclid's *Elements* present the first extant systematic exposition of geometry based on definitions and axioms. The definitions are placed at the beginning of most of the thirteen 'books' constituting the *Elements*. Thus, at the beginning of Book I Euclid offers definitions of the various objects of planimetry:

> 1. A *point* is that which has no part. 2. A *line* is breadthless length… 4. A *straight line* is a line which lies evenly with the points on itself. 5. A *surface* is that which has length and breadth only… 7. A *plane surface* is a surface which lies evenly with the straight lines on itself.
>
> (Euclid, *Thirteen Books,* vol. 1, p. 153)

Euclid also defines an angle and its types, a plane figure, a circle and its centre and diameter, a polygon, the types of triangles and quadrangles and parallel lines.

Book I continues by listing the axioms among which Euclid distinguishes 'postulates' and 'general concepts'. It is the latter ones that are often called axioms.[1] Postulates express the main properties of geometric constructions accomplished by a perfect ruler and compasses. The first two postulates claim that it is possible to draw a straight line between any two points and to produce any such line indefinitely. Postulate III states that a circle might be drawn with any point as its centre and with any radius. According to postulate IV, all right angles are congruent. Postulate V, which underlies the theory of parallel lines (see later), is the most involved one. It reads thus:

> If a straight line [EF on Figure 14.8] falling on two straight lines [AB and CD] situated in the same plane [with EF], makes the interior angles [BGH and GHD] on the same side together less than two right angles, the two straight lines [AB and CD], if produced infinitely, meet on that side [BD] on which the angles are together less than two right angles.
>
> (Euclid, *Thirteen Books,* vol. 1, p. 155)

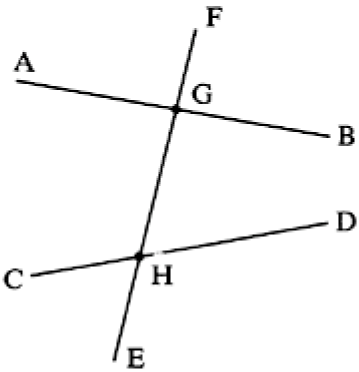


Figure 14.8

The 'general concepts', or axioms proper, make it possible to compare quantities with each other. These axioms are as follows:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

(Euclid, *Thirteen Books,* vol. 1, p. 155)

According to the standpoint of our time, all this system of premisses is still insufficient for constructing the geometry of the usual space, i.e. of the space described in Euclid's *Elements* and therefore called Euclidean. Only at the turn of the nineteenth century did mathematicians create complete systems of axioms for this space. This happened after the total revision of all the systems of the Euclidean premisses which, in turn, was occasioned by the discovery of the Lobachevskian hyperbolic geometry, where all the axioms of the Euclidean space except postulate V are fulfilled.

However, a critical analysis of Euclidean definitions and axioms goes back many centuries. Thus, for example, Arab scholars developed a general theory of ratios and proportions intending to replace the one described in Book V of the *Elements*. Many antique and medieval scientists turned their main attention to postulate V since Euclid had formulated it in a very complicated way and, moreover, had proved its reciprocal proposition (proposition 28 in Book I of the *Elements*) without recourse to the postulate. Even antique authors, prompted by the complexity and the non-obviousness of postulate V, attempted to prove it as a theorem. Below, we shall discuss such attempts made in antiquity and in Arabic mathematics noting that one of the Arab mathematicians interested in this problem, Naṣīr al-Dīn al-Ṭūsī (1201–1274), considered that an even more radical revision of the systems of 'general concepts' and postulates was necessary. At the beginning of his *Exposition of Euclid* (*Taḥrīr Uqlīdis*) he writes:

> I say, first of all, of what is necessary: it is supposed that the point, the line, the surface, the straight line, the plane surface, and the circle exist and that we

choose a point on any line or surface and we assume a line on any surface or passing through any point.

(al-Ṭūsī, *Taḥrīr Uqlīdis*, p. 3)

Thus, al-Ṭūsī suggested to supplement the Euclidean system of initial premisses by new axioms on the existence of points, lines, straight lines and of the other geometric figures defined by Euclid in the opening lines of Book I of the *Elements*.

Al-Ṭūsī's ideas were developed in the *Book of Exposition of the Euclidean Elements* (*Kitāb Taḥrīr al-Uṣūl li-Uqlīdis*) published in Arabic (Rome, 1594) under the name of this scientist. Actually, however, its author completed the book in 1298, twenty-four years after al-Ṭūsī's death. It is certain that the author belonged to al-Ṭūsī's school and apparently he was one of his students. Furthermore, he seems to be the son of al-Ṭūsī, Ṣadr al-Dīn, who, after his father's death, took charge of the Marāgha observatory. It is possible that in preparing the Roman edition the scribes who rewrote the original manuscript, owing to the great popularity still enjoyed by Naṣīr al-Dīn al-Ṭūsī, unintentionally omitted the first two parts of names of its real author, Ṣadr al-Dīn ibn Khwāja Naṣīr al-Dīn al-Ṭūsī. Having ascertained that this writing was completed after al-Ṭūsī had died, scholars usually call it *Pseudo-Ṭūsī's Exposition of Euclid*.

Unlike the *Taḥrīr Uqlīdis* of al-Ṭūsī himself, this book expressly formulates the axioms of existence of geometric objects and regards these axioms as new postulates. Then the proofs of all the Euclidean postulates are offered. (We comment on the proof of postulate V in the section The theory of parallel lines'.) Both the postulates of existence and the proofs of the Euclidean postulates are also included in the geometric part of *The Pearl in the Crown for Adoring Dībāj (Durra al-Tāj li-Ghurra al-Dī bāj),* an encyclopedic writing by al-Ṭūsī's pupil Quṭb al-Dīn al-Shīrāzī (1236–1311).

Ibn al-Haytham, in his two books devoted to commenting on Euclid's *Elements, The Book of Commentaries on the Introductions in Euclid's Elements* (*Kitāb sharḥ muṣādarāt kitāb Uqlīdis fī al-Uṣūl*) and *The Book on the Resolution of Doubts in Euclid's Elements and Interpretation of its Meanings* (*Kitāb fī ḥall shukūk kitāb Uqlīdis fī al-Uṣūl wa-l-sharḥ ma'ānīhi*), was the first Arabic mathematician to formulate the problem of the existence of geometric objects. In the second book, referring to the first one, he wrote: 'We have ascertained, in the *Commentaries on the introductions,* that there exist in mathematics such quantities as solids, surfaces, and lines; they exist in the mind's eye and this existence takes place by abstracting from sensed bodies (Ibn al-Haytham, *Kitāb fī ḥall Shukūk Uqlīdis*, p. 7). He also maintained that 'Speculation about the existence of things is the business of philosophers rather than mathematicians' (*ibid*.: 6) and he continued:

Existing things are subdivided into two types: those existing in the senses, and those existing in the imagination and in abstraction, but that which exists in the

> senses does not exist in truth since senses often err without detection by the observer…whereas that which exists in the imagination exists in truth and absolutely because the form that shapes itself in the imagination is real since it does not disappear or change.
>
> (*Ibid*.: 20–1)

THE THEORY OF PARALLEL LINES

Investigations on the theory of parallel lines which attempted to prove the relevant Euclidean postulate played an especially important role in the history of geometry. The comparative complexity of the formulation of this postulate (see the previous section) perhaps testifies that it was added to the other postulates at some later date. At any rate, this postulate or some equivalent statement is necessary in order to prove many theorems on triangles contained in Book I of the *Elements,* as well as the Pythagorean proposition that crowns Book I and, for that matter, is indispensable for the entire theory of similarity explicated in Book VI. Apparently even Euclid's forerunners in the fourth century BC searched for a more obvious and convincing axiom to serve as the basis of the theory of parallel lines.

It is possible to conclude from Aristotle (*Works,* vol. 9, p. 65a) that during his lifetime or even earlier some scholars attempted to prove one or another proposition equivalent to postulate V. That Aristotle himself offered a certain version of such a proposition is not impossible. In any case, ʻUmar al-Khayyām, in his *Commentaries on the Difficulties [encountered] in the Introductions to the Book of Euclid* (*Sharḥ mā ashkāla min muṣādarāt kitāb Uqlīdis* wrote that 'the cause of the mistake made by subsequent scientists in proving this premise [the Euclidean postulate V] was that they did not take into account the principles borrowed from the Philosopher [from Aristotle]' (al-Khayyām, *Rasā'il* (Traktaty), pp. 119–20. Khayyām went on to formulate five such principles:

> (1) Quantities might be divided up to infinity, that is, they do not consist of indivisibles; (2) A straight line might be produced to infinity; (3) Every two intersecting straight lines open up and diverge as they move away from the apex of the angle of the intersection; (4) Two convergent straight lines intersect and it is impossible for two convergent straight lines to diverge in the direction in which they converge; (5) Of two unequal limited quantities the smaller might be taken with such multiplicity that it will surpass the greater one.
>
> (*Ibid*.: 41–2)

Below we shall discuss Aristotle's statement equivalent to principle 1. We agree that assertions tantamount to principles 2, 3 and 5 are also contained in his works. Principle 4, or, rather, each of its two statements, is equivalent to Euclid's postulate V and it is possible that Aristotle had put forward this principle in a writing which did not reach our time. Whereas in postulate V the condition that two straight lines intersect when being produced is that the sum of the interior angles on the same side (angles BGF and EHD in Figure 14.8) is less than two right angles, in principle 4 the corresponding condition is that lines AB and CD approach each other in the direction of B (or D).

As far as we know, the first post-Euclidean writing devoted to the theory of parallel lines was the lost Archimedes's treatise *On Parallel Lines*. The Arab historian al-Qifṭī mentioned it under the title *Kitāb al-khuṭūṭ al-mutawāziyya* along with other writings of this scholar then available in Arabic translations. Posidonius (second-first centuries BC), Ptolemy (second century), Proclus (fifth century), Aghānis and Simplicius (fifth-sixth centuries) attempted to prove postulate V. Aghānis's proof is extant in the Arab mathematician al-Nayrīzī's (d. 922) commentary on Euclid's *Elements*. Both Posidonius and Aghānis started from defining parallel lines as lines lying in the same plane and being at a constant distance from one another. (According to Euclid, parallel lines do not intersect in their common plane when produced in either direction.)

Since the possibility that such lines exist follows from postulate V and some other axioms of the Euclidean geometry, these attempts to prove the postulate had to use implicitly a proposition equivalent to it.

In the Arabic East, ʿAbbās al-Jawharī, a contemporary of al-Khwārizmī, apparently made the earliest attack on postulate V. In his *Improvement of the Book 'Elements' (Iṣlāḥ li-kitāb al-Uṣūl)* he assumed that it was possible through any point situated inside an angle to draw a line intersecting both its sides. Many later geometers used this statement for proving postulate V. Actually however, the statement is tantamount to it and could not be proved by means of the other Euclidean axioms.

A few decades after al-Jawharī had made his attempt Thābit ibn Qurra offered two essentially different proofs of postulate V. One of them is contained in his *Book Showing that if a Straight Line Falls on Two Straight Lines in such a Way that the [Interior] Angles on the Same Side are Less than Two Right Angles, then these Straight Lines will Meet if Produced (Kitāb fī annahu idhā waqaʿa khaṭṭ mustaqīm ʿalā khaṭṭayn mustaqīmayn fa-sayyara al-zāwiyatayn allatayn fī jiha wāhida aqall min qāʾimatayn fainna al-khaṭṭayn idhā ukhrija fī tilka al-jiha iltaqayā).* (Some manuscript copies of this writing have a simple title *Book on the Proof of the Well-known Euclidean Postulate (Maqāla fī burhān al-muṣādara al-mashhūra min Uqlīdis)*.) The other proof is in Ibn Qurra's *Book Showing that Two Lines Drawn at Angles Less than Two Right Angles to Each Other will Meet (Maqāla fīanna al-khaṭṭayn idhā ukhrija ilā al-zāwiyatayn aqall min al- qāʾimatayn iltaqayā).*

His first proof is based on the assumption that, if two lines intersected by a third one approach, or move away from each other when produced in one direction, they move away from or approach each other when produced in the opposite direction.

By means of this assertion Ibn Qurra (see Houzel 1991) proves that a parallelogram exists and goes on to derive postulate V. Nowadays we know that, according to the Lobachevskian hyperbolic geometry where this postulate does not hold (though the other axioms of the Euclidean system are still valid), there exist 'divergent lines' which diverge from each other in both directions from their common perpendicular. On the contrary, within the boundaries of the Riemannian elliptic geometry where postulate V is fulfilled but some other axioms of the Euclidean geometry are dropped, any two straight lines approach each other and intersect, again in either direction from their common perpendicular.

In his second work Thābit ibn Qurra commences from a quite different assumption. Considering a ‘simple movement’, i.e. a uniform translational movement along a certain straight line (a parallel translation) of a certain body (e.g. of a straight segment perpendicular to the line), he holds that all points of the body (of the segment) describe straight lines. Ibn Qurra inferred that equidistant straight lines do exist. Actually, however, his assumption is only true in the Euclidean geometry, whereas according to the Lobachevskian hyperbolic geometry the points moving translationally along a straight line describe arcs of curved lines, the so-called equidistants, or loci situated at an equal distance from the straight lines.

Being guided by his assumptions Thābit ibn Qurra proves that a rectangle exists, hence deriving postulate V. Note that the Syriac historian and astronomer Ibn al-ʿIbrī, alias Bar Hebraeus (1226–1286), while listing in his *Chronography* the Syriac writings of Thābit ibn Qurra, mentioned both his treatises on parallel lines (Bar Hebraeus, *Gregorii Abulpharagii,* p. 180).

Ibn al-Haytham offered an original derivation of postulate V in his *Commentaries on the Introductions to the Books of Euclid*. He begins by considering the movement of a perpendicular to a straight line. Basing himself on the same ‘simple movement’ as Thābit ibn Qurra did, Ibn al-Haytham proved that the end-point of the ‘perpendicular whose foot remains on the line describes a straight line. He states that all points of the perpendicular describe ‘similar and equal lines’ and that since its foot moves along a straight line its other end-point does the same. Recall, however (see earlier), that the assumption that all points describe similar and equal lines while moving translationally along a straight line, is tantamount to the Euclidean postulate V.

The introduction of a quadrangle with three right angles originated with Ibn al-Haytham. Later, J.H.Lambert, whom we mentioned earlier, once more considered such a quadrangle in attempting to prove postulate V. The fourth angle of the ‘Lambert quadrangle’ might be acute, obtuse or right (Figure 14.9). Ibn al-Haytham refuted the existence of the two first possibilities by using his theorem that the end-point of the moving perpendicular describes a straight line. After proving the existence of the quadrangle, he quite simply derived postulate V. Actually, the two rejected hypotheses are theorems of the hyperbolic and the elliptic geometries respectively.

We especially note that, in proving that a perpendicular and an inclined line drawn to the same line intersect each other, Ibn al-Haytham formulated an important assumption, considering it as self-evident. In 1882, the German geometer M.Pasch introduced this assumption anew as an axiom of order: a straight line lying in the plane of a triangle and meeting one of its sides, he maintained, if accurately produced will either meet a second side of the triangle or pass through the apex opposite to the first side. Naṣīr al-Dīn al-Ṭūsī directly used the same proposition in his theory of parallel lines.

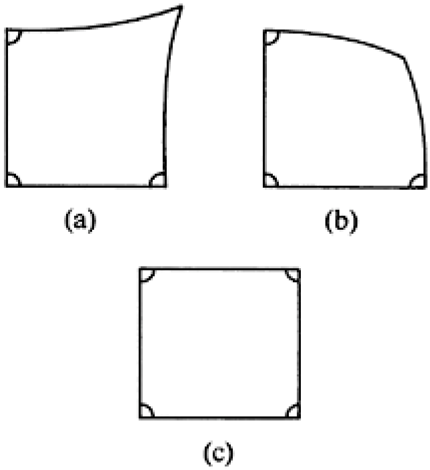


Figure 14.9

Thus, in attempting to prove postulate V, both Thābit ibn Qurra and Ibn al-Haytham as well as their predecessors had indeed committed the logical mistake of *petitio principi* as mentioned by Aristotle.

Ibn al-Haytham also touched on the theory of parallel lines in his second work devoted to commenting on the *Elements,* i.e. in his *Book on the Resolution of Doubts in Euclid's 'Elements' and on Interpretation of its Meanings*. Here, however, he restricted himself to referring to his first book and to a remark that it was possible to replace postulate V 'by another one which is more obvious and which penetrates deeper into the soul, namely: any two intersecting straight lines cannot be parallel to one and the same line' (Ibn al-Haytham, *Kitāb fī* *ḥall,* p. 25).

ʿUmar Khayyām, in his Book I of *Commentaries on the Difficulties [Encountered] in the Introductions to the Book of Euclid,* criticized Ibn al-Haytham's proof and replaced it by another one. Khayyām rejected the use of movements in geometry and proved postulate V based on another explicitly assumed postulate which he considered more simple, namely, on the fourth of the five 'principles due to the Philosopher' (to Aristotle). Thus, Khayyām avoided the logical mistake made by his forerunners. He then considered a quadrangle with two right angles at its base and equal lateral sides and studied the three suppositions about the two remaining equal angles (Figure 14.10). (G.Saccheri (1667–1733) introduced the same quadrangle in his theory of parallel lines and the figure is often named after this Italian mathematician.) By means of his principle mentioned above Khayyām refutes the hypotheses that the angles are either acute or obtuse and proves postulate V.

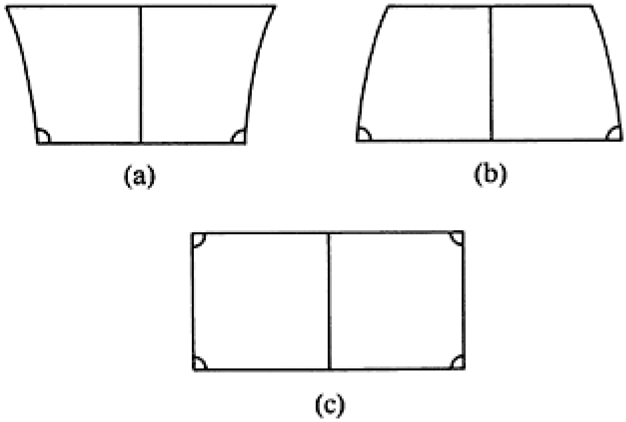


Figure 14.10

Al-Bīrūnī also engaged in the theory of parallel lines. In the list of his writings which he himself compiled there is a *Book Showing that the Property of Quantities to be Divided to Infinity is Similar to the Situation with Two Straight Lines Approaching Each Other but not Meeting when Moving off (Maqāla fī anna lawāzim tajzī' al-maqādir ilā lā nihāya qarība min amr al-khaṭṭayn alladhayn yuqrabān wa lā yultaqiyān fī al-istib'ād*. A recently discovered fragment of al-Bīrūnī's treatise contains the reasoning of Ya'qūb al-Kindī, who, based on the existence of parallel lines, proved that quantities may be divided to infinity; the fragment also includes the author's own thoughts on the subject and it therefore apparently belongs to this treatise. Since Khayyām, in proving postulate V, uses both the fourth and the first principle of Aristotle on the strength of which quantities are divisible to infinity, it is reasonable to infer that Khayyām was acquainted with the mentioned works of al-Kindī and al-Bīrūnī.

In turn, Ḥusām al-Dīn al-Sālār (d. 1262) certainly read Khayyām's treatise. He worked at first in Khwārizm and, after the Mongols' conquest of this country, continued at the courts of Chingiz-Khān and of his successors, including Hūlāgū-Khān. Al-Sālār wrote *Premisses for the Proof of the Postulate Introduced by Euclid at the Beginning of his First Book Concerning Parallel Lines (Muqaddimāt li-tabyīn al-muṣādara allatī dhakaraha Uqlīdis fī ṣadr al-maqāla al-ūlā fī mā yata'allaqu bi-l-khuṭūṭ al-mutawāziya).* That he was familiar with Khayyām's treatise is testified not only by his lame attempt to prove postulate V (where he made a glaring mistake), but also by his proof of Aristotle's principle 3 used by Khayyām.

Naṣīr al-Dīn al-Ṭūsī was also acquainted with Khayyām's treatise and, in addition, he perhaps knew al-Sālār's writing. Together with al-Sālār he worked at the astronomical observatory in Marāgha under Hūlāgū-Khān's court. Naṣīr al-Dīn al-Ṭūsī considered the theory of parallel lines in two works, namely, in the *Treatise that Cures Doubts in Parallel Lines (al-Risāla al-shāfiya 'an shakk fī al-khuṭūṭ al-mutawāziyya)* specially devoted to the theory, and in the *Exposition of Euclid,* this being, indeed, an exposition of Euclid's *Elements* with considerable additions made by the author. In both instances, al-Ṭūsī, like Khayyām, considered the 'Saccheri quadrangle' and investigated the three hypotheses concerning its upper angles. In the *Treatise that Cures Doubts* al-Ṭūsī, before offering his own proof of postulate V, expounded the theories of parallel lines developed by al-Jawharī, Ibn al-Haytham and Khayyām. He correctly indicated the weak point in al-Jawharī's proof. Al-Ṭūsī did not read Ibn al-Haytham's proof as given in the *Commentaries on the Introductions to the Book of Euclid.* He knew only the *Book on the Resolution of Doubts in Euclid's Elements* where he could have seen no more than a reference to the former work. Al- Ṭūsī knew therefore that Ibn al-Haytham had used movement while proving postulate V. However, he mistakenly inferred that the latter's proof was based on the statement that any two intersecting straight lines could not be parallel to a same line and he criticized Ibn al-Haytham for not deriving postulate V from this statement.

Neither did al-Ṭūsī know Khayyām's treatise in its entirety. Indeed, he described Khayyām's propositions without mentioning the five 'principles of the Philosopher' among which was the principle equivalent to postulate V. He blamed Khayyām for committing a logical error in proving this postulate. As we have seen above, this charge was unjust.

Al-Ṭūsī then went on to offer his own proof of postulate V. Some of his propositions, as he himself indicated, were borrowed from Khayyām. Again, he twice introduced the two last propositions of the proof taking their second version from al-Jawharī. Unlike Khayyām, al-Ṭūsī, in his *Treatise that Cures Doubts,* does not use a postulate equivalent to the Euclidean postulate V, and, just as most previous geometers, makes a mistake of *petitio principi*.

In a letter to al-Ṭūsī, ʿAlam al-Dīn Qayṣar al-Ḥanafī pointed out this mistake. As a result, al-Ṭūsī, while reproducing the proof of postulate V from the *Treatise that Cures Doubts* in his *Exposition of Euclid,* commenced by stating a postulate similar to but stronger than the one used by Khayyām. (Khayyām's postulate had excluded the case of the hyperbolic geometry whereas al-Ṭūsī's postulate ruled out both the hyperbolic and the elliptic geometries.) If straight lines lying in the same plane, reads al-Ṭūsī's postulate, diverge in one direction, they cannot converge in this direction if only they do not meet (al-Ṭūsī, *Taḥrīr Uqlīdis* p. 4).

In *Pseudo-Ṭūsī's Exposition of Euclid,* written by a member of al-Ṭūsī's school, another statement is used instead of a postulate. It was independent of the Euclidean postulate V and easy to prove. However, in the sequel Pseudo-Ṭūsī committed the error of *petitio principi*. He essentially revised both the Euclidean system of axioms and postulates and the proofs of many propositions from the *Elements*.

His book published in Rome considerably influenced the subsequent development of the theory of parallel lines. Indeed, J.Wallis (1616–1703) included a Latin translation of the proof of postulate V from this book in his own writing *On the Fifth Postulate and the Fifth Definition from Euclid's Book 6 (De Postulato Quinto et Definitione Quinta lib. 6 Euclidis,* 1663). Saccheri quoted this proof in his *Euclid Cleared of all Stains (Euclides ab omni naevo vindicates,* 1733). It seems possible that he borrowed the idea of considering the three hypotheses about the upper angles of the 'Saccheri quadrangle' from Pseudo-Ṭūsī. The latter inserted the exposition of this subject into his work, taking it from the writings of al- Ṭūsī and Khayyām.

Quṭb al-Dīn al-Shīrāzī supplied yet another proof of postulate V in the geometric part of his above-mentioned encyclopedic work. However, he, like many other scholars, made the mistake of *petitio principi*.

While expounding a number of subjects, and, in particular, while formulating his postulates, al-Shīrāzī came closer to *Pseudo-Ṭūsī's* *Exposition of Euclid* than to al-Ṭūsī's own work of the same title.

Thus, for at least four centuries, the theory of parallel lines attracted the attention of mathematicians of the Near and the Middle East whose writings reveal a direct continuity of ideas. Three scientists, Ibn al-Haytham, Khayyām and al-Ṭūsī, had made the most considerable contribution to this branch of geometry whose importance came to be completely recognized only in the nineteenth century. In essence their propositions concerning the properties of quadrangles which they considered assuming that some of the angles of these figures were acute or obtuse, embodied the first few theorems of the hyperbolic and the elliptic geometries. Their other proposals showed that various geometric statements were equivalent to the Euclidean postulate V. It is extremely important that these scholars established the mutual connection between this postulate and the sum of the angles of a triangle and a quadrangle.

By their works on the theory of parallel lines Arab mathematicians directly influenced the relevant investigations of their European counterparts. The first European attempt to prove the postulate on parallel lines made by Witelo, the Polish scientist of the thirteenth century, while revising Ibn al-Haytham's *Book of Optics* (*Kitāb al-manāẓir*)—was undoubtedly prompted by Arabic sources. The proofs put forward in the fourteenth century by the Jewish scholar Levi ben Gerson, who lived in Southern France, and by the above-mentioned Alfonso from Spain directly border on Ibn al-Haytham's demonstration.

Above, we have indicated that *Pseudo-Ṭūsī's* *Exposition of Euclid* had stimulated both J.Wallis's and G.Saccheri's studies of the theory of parallel lines. The coincidence of the approaches to the three hypotheses about the angles of a quadrangle by medieval Eastern Scholars on the one hand and by Saccheri and Lambert on the other is extremely interesting.

GEOMETRIC TRANSFORMATIONS

The problem of considering mechanical movement in geometry which we encountered while discussing the proofs of postulate V in the works of Thābit ibn Qurra, Ibn al-Haytham and Khayyām, originated even in antiquity. The use of movement and superposition had underlain all the proofs of the theorems due to Thales during whose lifetime the Euclidean axioms and postulates were not yet formulated. The Pythagoreans used movement to the same degree. They regarded a line as a trace of a moving point, and accordingly, they believed that a surface was a trace of a moving line.

However, Aristotle objected to the use of movement in theoretical mathematics and Euclid obviously attempted to minimize the number of instances when figures were superposed one upon another, but in spite of his efforts he failed to exclude superposition altogether. Aristotle substantiated his opinion by stating that a point was an abstraction of a higher degree than a line; that a line was more abstract than a surface; and that a surface was a more abstract notion than a body. He considered it expedient to obtain abstractions of lower degrees from those of higher degrees.

Al-Fārābī was greatly influenced by Aristotle. In his *Commentaries on the Difficulties [Encountered] in the Introductions to the First and Fifth Books of Euclid* (*Sharḥ al-mustaghlaq min muṣādara min al-maqāla al-ūlā wa-l-khāmisa min Uqlīdis)* he adhered to the same idea. Considering the sequence in which Euclid formulated his definitions of the point, the line, the surface and the body, al-Fārābī indicated: 'Instruction should begin by considering the sensed body; it should then pass on to studying bodies abstracted from the sensations connected with it, then to surfaces, then to lines and points' (al-Fārābī, *Al-Rasā'il,* p. 239).

Remaining on Aristotle's standpoint, al-Fārābī analysed the other definitions contained in Books I and V of the *Elements*. Adhering to the same point of view, Khayyām found fault in Ibn al-Haytham's proof of postulate V:

> What relation is there between geometry and movement and what should be understood as movement? According to the views of scientists, it is doubtless that a line might only exist on a surface, and a surface in a body, that is, a line might only be in a body and cannot precede a surface. How, then, can it move while being abstracted from its subject? How can a line be formed by a movement of a point whereas it precedes the point by its essence and by its existence?
>
> (al-Khayyām, *Rasā'il (Traktaty),* pp. 38, 115)

Despite such criticisms, many subsequent scientists used movement while solving geometric problems. In particular, Levi ben Gerson and Alfonso reasoned in terms of movement when proving postulate V and, also, while considering other subjects.

In addition to movement, antique mathematicians used more general geometric transformations. Thus Democritus's reasoning that pyramids with equal bases and heights have the same volumes was founded on a particular case of an affine transformation, namely, or a *shift* with all points of the base of one pyramid remaining fixed and the sections parallel to the base being shifted in proportion to their distance from the base.

Archimedes, in his work *On Spheroids and Conoids,* calculated the area of an ellipse by using another special affine transformation, namely, a *contraction* of a circle to one of its diameters.

Apollonius considered yet another affine transformation, a *homothety* (a central similarity) and an inversion in a circle, in his treatise *On Plane Loci*. A homothety is a transformation of a plane under which any of its points M shift onto point M′ lying on the straight line M_0M with $M_0M'=kM_0M$, M_0 being *the centre of the homothety* and *k* its *coefficient* (Figure 14.11). Under inversion in a circle any point M of a plane shifts onto point M′ of the straight line M_0M and $M_0M'=r^2/M_0M$ with M_0 being *the centre of inversion* and *r* the *radius of the circle of inversion* (Figure 14.12). Under homothety straight lines shift onto straight lines and circles shift onto circles; under inversion straight lines and circles shift onto circles but both the straight lines and the circles passing through the centre of inversion become straight lines.

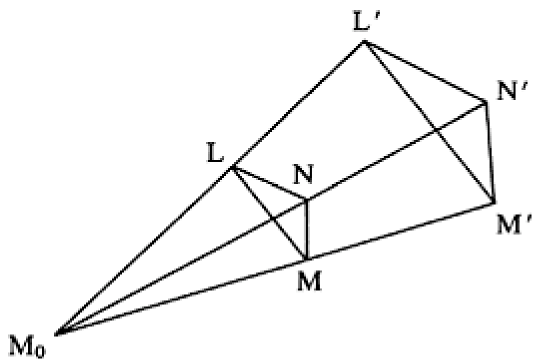


Figure 14.11

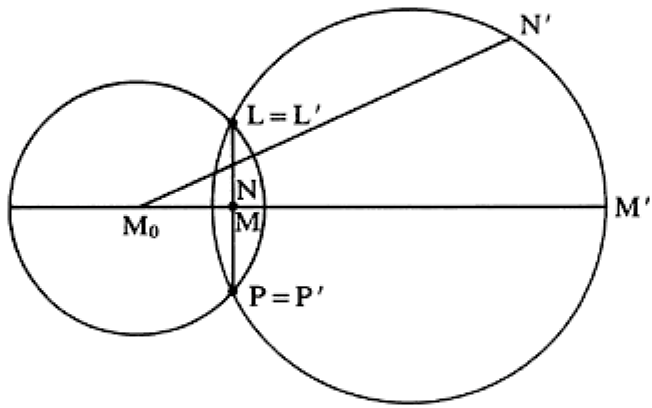


Figure 14.12

Apollonius knew all these facts and indicated that the plane loci, this being the Greek term denoting straight lines and circles together, pass to plane loci. In proposition I_{37} of his *Conics,* Apollonius actually considered inversions not only in a circle, but, in addition, in an ellipse and in a hyperbola, i.e. transformations of points M on a given plane to points M′ of intersection or their polar lines with the diameter of the respective conic section through M. In propositions I_{33} and I_{35} he considered a similar inversion in a parabola.

Affine transformations of a plane or of a space are such transformations of these objects under which straight lines pass to straight lines (owing to the bijectiveness of these transformations parallel lines pass to parallel lines). Particular cases of the affine transformations are motions; shifts used by Democritus; direct contractions or dilatations used by Archimedes; and oblique contractions or dilatations under which points move along straight lines not perpendicular to the fixed axis or to the fixed plane, and homothety. The ratios of the areas of plane figures and of the volumes of solids are preserved under any affine transformation. When, furthermore, the areas or volumes themselves remain unchanged, as, for example, with motions and shifts, the corresponding affine transformations are called *equiaffine*.

Thābit ibn Qurra and his grandson Ibrāhīm ibn Sinān used both general affine and equiaffine transformations. The latter, in his *Book on Constructing the Three Sections (Maqāla fī rasm* **al-quṭūʿ** *al-thalātha),* constructed ellipses by a direct contraction to circles. He also constructed equilateral and arbitrary hyperbolas obtaining several points lying on these sections from corresponding points of a circle (arbitrary hyperbolas can also be obtained by direct contractions to equilateral hyperbolas).

Thābit ibn Qurra, in his *Book on the Sections of the Cylinder and on their Surface (Kitāb fī* **quṭūʿ al-usṭuwāna** *wa* **basīṭihā**) considered equiaffine transformations

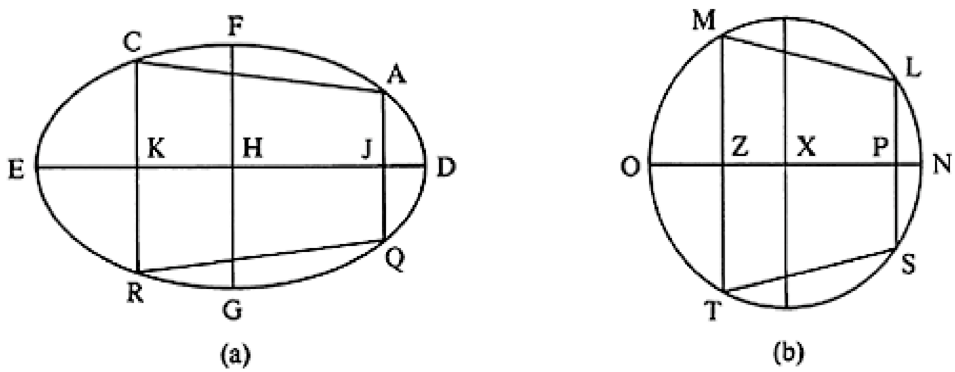


Figure 14.13

that passed an ellipse with semiaxes *a* and *b* into a circle with radius $\sqrt{(ab)}$. He proved that under this transformation the segments of an ellipse become equiareal segments of the corresponding circle. Figure 14.13 reproduces one of his drawings with which he illustrated this theorem.

Finally, we note that Ibrāhīm ibn Sinān, in his *Book on Measuring the Parabola (Kitāb fī* **misāḥa al-qaṭʿ** *al-mukāfī),* used an arbitrary affine transformation of polygons and segments of parabolas. In proposition 1 he considered two polygons, ABCDE and GHJIK, obtained one from the other by an affine transformation (Figure 14.14) and proved that the ratio of the area of the first polygon to the area of the second one is equal to the ratio of the areas of triangles ADE and IKG inscribed in them.

In his proposition 2, Ibn Sinān extended this statement onto segments of parabolas proving by the method of exhaustion that the ratios of the areas of the segments ABC (Figure 14.15(a)) are equal to the ratios of the areas of triangles having bases AC and 'apexes' B of the segments as their own bases and 'apexes' (by apexes of a segment we mean the end-points of the diameter conjugate to the base of the segment).

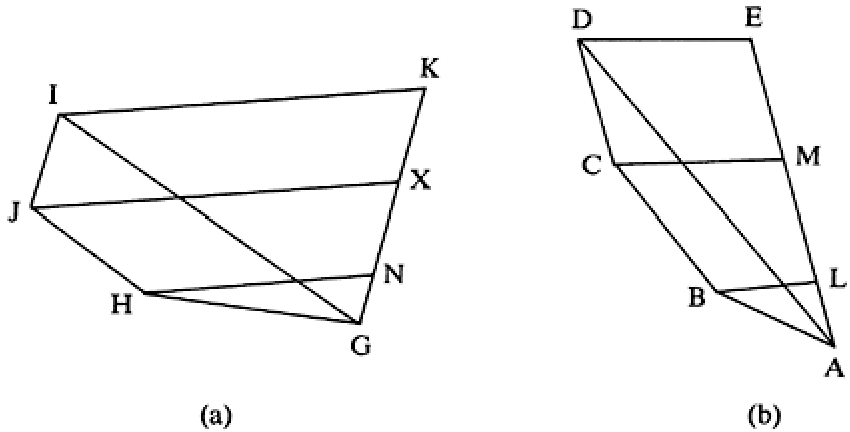


Figure 14.14

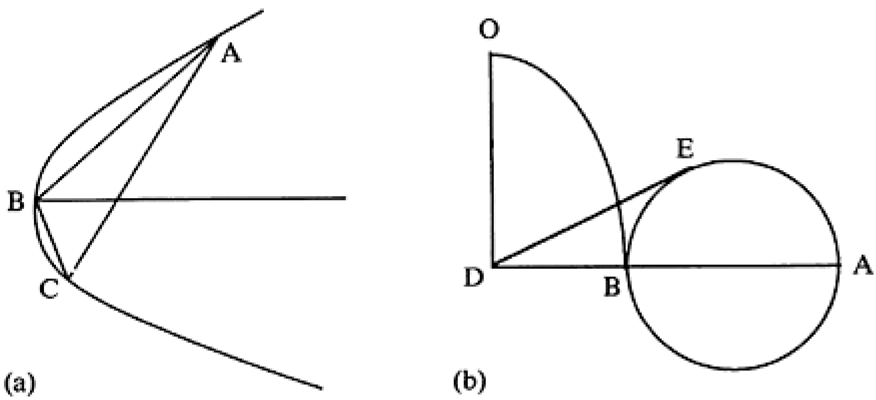


Figure 14.15

Recently Irina O.Luther and Sadiqjān A.Vahabov have discovered projective transformations of a circle into conic sections in the works of Ibrāhīm ibn Sinān and al-Bīrūnī. Ibrāhīm ibn Sinān, in his *Book on the Constructing of Three Sections,* proposes the construction of an equilateral hyperbola 'with the aid of a circle' as follows: at an arbitrary point E of the circle AEB (Figure 14.15(b)) he draws the tangent ED to the diameter AB of the circle and raises in D the perpendicular DO=ED to the diameter AB, the point O is a point of the hyperbola. If the equation of the circle is $x^2+y^2=a^2$ and the diameter AB is the abscissa, the equation of the obtained hyperbola is $x^2-y^2=a^2$, this

$$x' = \frac{a^2}{x} \qquad y' = \frac{ay}{x}$$

being involutive homology with the centre A and axis tangent to the circle in B. (See Ibn Sinān, *On Drawing Conics,* pp. 39–40.)

In changing the perpendicular DO=ED by the lines of the same length drawn under the constant angle Ibn Sinān obtains a common hyperbola, this hyperbola has the same equation but in oblique angle co-ordinates. To obtain a common hyperbola from the equilateral one Ibrāhīm ibn Sinān uses also the contraction of this hyperbola to the diameter AB analogous to contraction of the circle to an ellipse used by him in the same book.

Al-Fārābī and Abū al-Wafā' offered a number of constructions actually based on homothety. Al-Kūhī devoted one of his *Two Geometric Problems (Mas' alatān handasiyyatān)* to proving that under this transformation circles pass to circles.

After the tenth century, geometric transformations apart from those needed in constructing astrolabes and other astronomical instruments became considerably less interesting. In Europe, general affine transformation first appeared in the eighteenth century, in the works of A.C.Clairaut and L.Euler. During the next century, the theory of these transformations, and of more general projective transformations of planes and spaces, as well as the theories of inversive (Möbius's) transformations of the plane and the space, were created, (These transformations are generated by inversions in circles and in spheres.)

PROJECTIONS

Even ancient Greeks were familiar with the projections of one surface (or plane) onto another underlying the concept of the just mentioned projective transformation. The Roman architect Vitruvius (first century) indicated three types of projections that were used in his time: the horizontal and the vertical projections of buildings (*ichnography* and *orthography*) and the perspective images shown on sceneries in the theatres *(scenography)*.

Diodorus (first century BC), in his *Analemma,* orthogonally projected the celestial sphere onto a plane, as did Ptolemy in a work of the same title. The geographical writings of Eratosthenes, Marinus of Tyre and Ptolemy contain various projections of the inhabited part of the Earth onto a plane.

In Ptolemy's *Planisphaerium* we find the stereographic projection of a sphere onto a plane, i.e. a projection of a sphere from one of its points either onto a plane tangential to the sphere at the point opposite to the chosen one (Figure 14.16), or onto a plane parallel to it. Ptolemy may have known that circles passing through the centre of this projection were represented by straight lines while the other circles of the sphere were represented by circles. Incidentally, this can be proved by proposition I_5 of Apollonius's *Conics* concerning two sets of circular sections of an oblique circular cone and it is possible that Apollonius himself was familiar with this property of the stereographic projection.

Arab mathematicians followed suit in systematically depicting solid figures using the parallel and, in particular, the orthogonal projection. Ḥabash al-Ḥāsib (*c.* 870–*c.* 970) and al-Bīrūnī were familiar with Diodorus's methods described in his *Analemma* and used them for determining the direction of the *qibla* (the direction towards Mecca to which Muslims should turn their faces while praying). Al-Bīrūnī expounded the appropriate work of Ḥabash al-Ḥāsib in a special writing addressed to his friend Abū Sa'īd al-Sijzī. In addition, he offered his solutions of the problem in his *Book on Determining the Boundaries of Places for Defining More Exactly the Distances between Settlements (Kitāb taḥdīd nihāyāt al-amākin li-taṣḥīḥ masāfāt al-masākin),* which is usually called *Geodesy,* and, again, in his *Canon Masudicus (al-Qānūn* Ibn

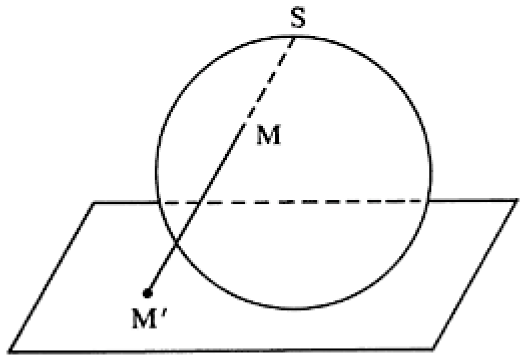


Figure 14.16

al-Mas'ūdī) al-Haytham gave a similar solution of this problem in his *Reasoning on the Determination of the Azimuth of the Qibla (Qawl fī istikhrāj samt al-qibla).*

We shall describe al-Bīrūnī's train of thought from his *Canon Masudicus,* extremely interesting from the point of view of its geometric methods. In essence, al-Bīrūnī's solution consists in determining the zenith of Mecca on the celestial sphere and constructing its orthogonal projection onto the plane of the horizon of the given town. Then the straight line connecting this point with the centre of the horizon circle, i.e. the orthogonal projection of the town's zenith onto the plane of this circle, will indicate the direction of the qibla for that town.

Before giving the solution proper, al-Bīrūnī carries out the following mental construction on the celestial sphere. Let AZC be the horizon circle of the town with E being its centre; also, let diameter AEC of the circle be the meridian line with A and C being the points of the South and the North respectively so that ABC is a half of the circle of the meridian lying in the plane perpendicular to the plane of the horizon (Figure 14.17). Marking off arc CF equal to the latitude of the town on the circle of the meridian, we determine point F, the pole of the universe. If, further, arc FG equal to the Mecca's co-latitude is marked off along the same circle then point G will belong to the diurnal circle of Mecca's zenith. The centre of this circle, point K, is the foot of the perpendicular dropped from G on the diameter EF of the sphere. Mentally constructing the diurnal circle GHD, al-Bīrūnī determines Mecca's zenith H as the end-point of the radius KH of this circle parallel to the radius of the celestial equator whose angular distance from the meridian line is equal to the difference between the longitudes of the given town and Mecca.[2]

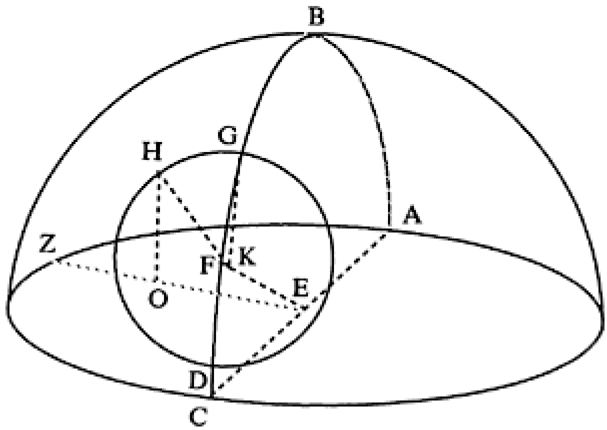


Figure 14.17

After determining Mecca's zenith, al-Bīrūnī constructs its orthogonal projection O onto the plane of the town's horizon and obtains the direction EOZ towards Mecca.

On al-Bīrūnī's drawing (Figure 14.18) the circles of the meridian and the celestial equator are rotated about the axis AC and superposed with the circle of the horizon. In addition, al-Bīrūnī rotated the half of the diurnal circle GHD which is parallel to the celestial equator about the axis GD so that this half became parallel to the plane of the horizon circle. Thus, al-Bīrūnī accomplished all his constructions in one single plane.

In his *Exhaustive Treatise on Shadows (Kitāb fī ifrād al-maqāl fī amr al- aẓlāl)* al-Bīrūnī once more superposes several planes. In this work he also described the main results of Diodorus's *Analemma*. Since the stereographic projection was used in constructing astrolabes it enjoyed great popularity in the Arab world. In his *Planisphaerium,* extant in an Arabic translation, Ptolemy failed to prove that under this projection the circles that did not pass through its centre were projected as circles. Aḥmad al-Farghānī (d. 861) supplied such a proof in his *Book on Constructing Astrolabes (Kitāb ṣan'at al-asṭurlāb)*. Subsequent scientists introduced other proofs of this most important property of the stereographic projection and Ibrāhīm ibn Sinān, while offering such a proof in his *Treatise on the Astrolabe (Risāla fī al-asṭurlāb),* referred to proposition I_5 of Apollonius's *Conics*.

We shall describe one more of al-Bīrūnī's determinations of the direction of the *qibla* from his *Book on Transfer of the Potency of Astrolabe to Actuality (Kitāb fī ikhrāj mā fī quwwa al-asṭurlāb ilā al-fi'l)*. Here, the author used another important property of the stereographic projection, namely, its conformity (the angles between lines on the sphere are equal to the angles between the projections of these lines onto the plane).

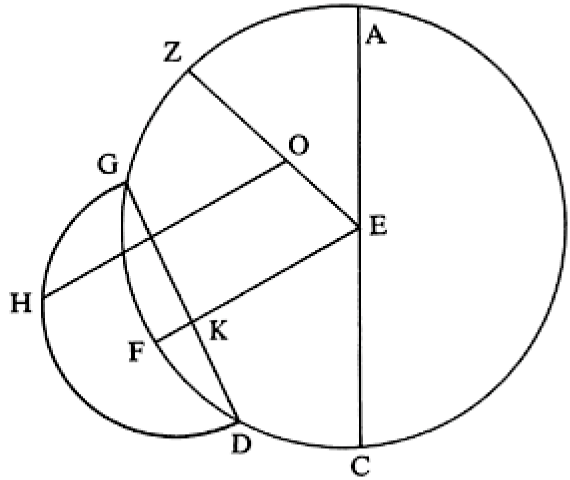


Figure 14.18

Al-Bīrūnī considered the spherical triangle MPZ situated on the surface of the Earth. Its vertices were the given town (Z), Mecca (M) and the North Pole (P) (Figure 14.19). The angle PZM of the triangle is called the azimuth of the *qibla* and determining the *qibla's* direction is the same as calculating this angle. In triangle MPZ the sides PM and PZ are equal to the co-latitudes of the given town and Mecca and angle MPZ is the difference between the longitudes of these towns. Al-Bīrūnī replaces this triangle by a similar one situated on the celestial sphere with its vertices being the zeniths of Mecca and the given town and the Northern Pole of the universe (we shall denote these apexes by the same letters M, Z and P), and considers the stereographic projection of the celestial sphere from the Southern Pole of the universe on-to the plane tangential to the sphere at the Northern Pole P. Under this projection the sides PM and PZ of the new spherical triangle MPZ are represented by straight segments PM′ and PZ′, both of them issuing from point P of the plane (Figure 14.20). The third side MZ of the triangle passes to the arc M′Z′ of the 'circle of azimuth' so that for determining the azimuth of the *qibla* it only remains to measure the angle between the arc M′Z′ and the segment Z′P′.

ʿAbd al-Jabbār al-Kharaqī (d. 1158), who worked in Marw and Khwārizm, improved al-Bīrūnī's method in his writing *The Supreme in Comprehending the Subdivision of the Celestial Spheres (Muntahā al-idrāk fī taqāsīm al-aflāk).* Whereas al-Bīrūnī necessarily insisted that the lines of azimuths (the verticals) be engraved on the plates of astrolabes, al-Kharaqī's method did not demand such lines. Instead, al-Kharaqī

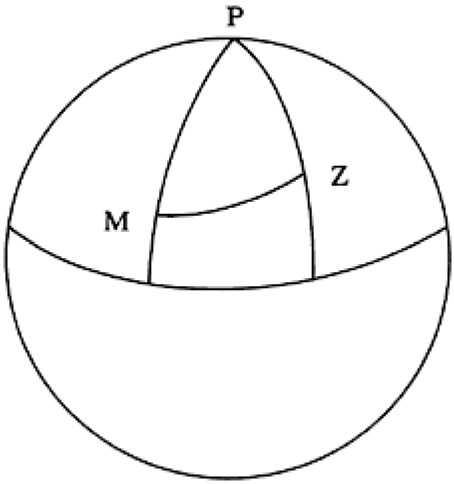


Figure 14.19

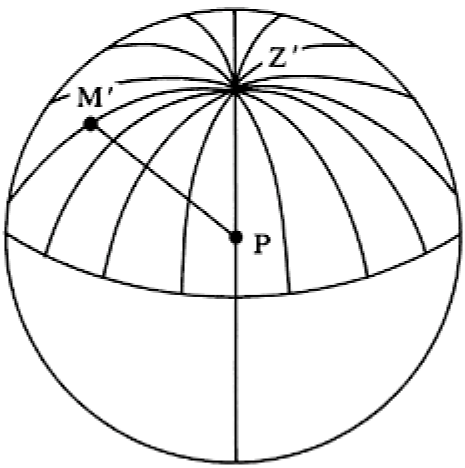


Figure 14.20

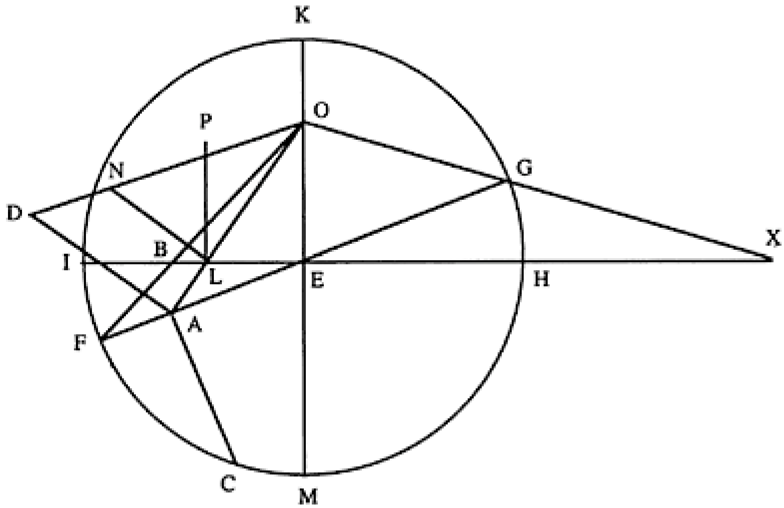


Figure 14.20(a)

had to make the astronomical observations at the moment when the Sun's altitude was equal to that of Mecca's zenith so that the azimuth of the *qibla* coincided with the hour angle (i.e. with the angle ZPS of the spherical triangle SPZ) and the shadow of the sundial's gnomon was directed at the *qibla*.

Maḥmūd al-Jaghmīnī (d. 1220), who worked in Khwārizm, explicated al-Kharaqī's method in his *Summary of Astronomy (al-Mulakhkhaṣ fī al-hay' a),* which proved extremely popular during the Middle Ages. Besides this, the method was described in numerous commentaries on this writing. Among the authors of these was Kamāl al-Dīn al-Turkumānī (fourteenth century) who worked in Saray, the capital of the Golden Horde. He expounded al-Kharaqī's method in great detail.

The stereographic projection is also used for mapping the surface of the Earth onto a plane, i.e. for the purpose of cartography. Since this projection is conformal, the angles between lines on the surface of the Earth are represented without distortions. Such maps are especially convenient for sailors.

Al-Bīrūnī devoted his *Treatise on Projecting the Constellations and on Representing Countries on Maps (Risāla fī tasṭīḥ al-ṣuwar wa tabṭīḥ al-kuwar)* to applying the stereographic projection to cartography. In the Arabic countries this projection was called 'the projection of the astrolabe' (*tasṭīḥ al-asṭurlāb*); at the beginning of the seventeenth century the Flemish physicist F.D'Aguillon introduced its modern name. L.Euler published two memoirs on the use of this projection in map compilation: applying analytical functions of a complex variable, he derived the general conformal representation of the Earth's surface by combining the stereographic projection with a conformal mapping of a plane onto itself.

In addition to the stereographic projection two other ones were used in constructing astrolabes, al-Ṣaghānī's 'perfect projection' *(al-tasṭīḥ al-tāmm)* and al-Bīrūnī's 'cylindric projection' of a sphere onto a plane. The first projection is the projection from a point not lying on the sphere onto a plane perpendicular to the straight line connecting the centres of the projection and of the sphere. The second projection is a parallel one. In both cases circles belonging to the sphere are in general represented by conic sections.

Al-Bīrūnī, in his *The Exhaustion of all Possible Methods of Construction of Astrolabe* (*Istīʿāb al-wujūh al-mumkina ṣanʿa al-asṭurlāb*), in the sections about al-Ṣaghānī's projection of a celestial sphere onto its equatorial plane from a point of its axis not being its pole (in which the circles of the sphere pass to conic sections), considers the construction of conic sections by means of projective transformation of a circle onto a conic section in its plane. Al-Bīrūnī considers the transformation of the circle KIMH onto the conic section KBM (Figure 14.20a) as follows: he sets a diameter FG of the circle KIMH and a point O on its diameter KM and for every point C of the circle he drops a perpendicular CA on the diameter FG, joins A with O, raises the perpendicular AD=AC to OA, joins D with O, draws the perpendicular LN from the point L of intersection of HI and OA to OD and raises the perpendicular LP=LN to HI. The point P is the point of the conic section through which passes the point C of the circle. If the angle MOG is acute, right and obtuse the conic section is respectively an ellipse, a parabola and a hyperbola. The points F and G of the circle pass respectively to points B and X of intersection of the lines OF and OG with the diameter HI (in the case of a parabola the lines OG and HI are parallel), the ends of the diameter of the circle perpendicular to FG pass through the points K and M. If the distance EO between the centre of the circle and the point O is equal to ϱ, and the angle FEI is equal to α, this projective transformation has the form

$$x' = \frac{\rho(x \cos\alpha + y \sin\alpha)\cos\alpha}{(x\cos\alpha + y\sin\alpha)\sin\alpha + \rho} \qquad y' = \frac{\rho(x\sin\alpha - y\cos\alpha)}{(x\cos\alpha + y\sin\alpha)\sin\alpha + \rho}$$

The constructed conic section is congruent to the central projection of the circle with diameter FG in the plane orthogonal to the plane of the drawing (this circle is the horizon in a town with latitude 90°−α) from the point O onto the equatorial plane of the sphere. Al-Bīrūnī describes also the analogous construction for *almucantars* of any altitude *h*—parallels of the horizon with spherical distance *h* from the horizon.

Recently, Roshdi Rashed has discovered 'conical' and cylindric projections in the writings of al-Qūhī and Ibn Sahl on astrolabes (Rashed 1993a).

Among other writings on astrolabes we note the work of Muḥyī al-Dīn al-Maghribī (d. *c.* 1290), an employee of the Marāgha observatory, *The Projection of the Astrolabe*

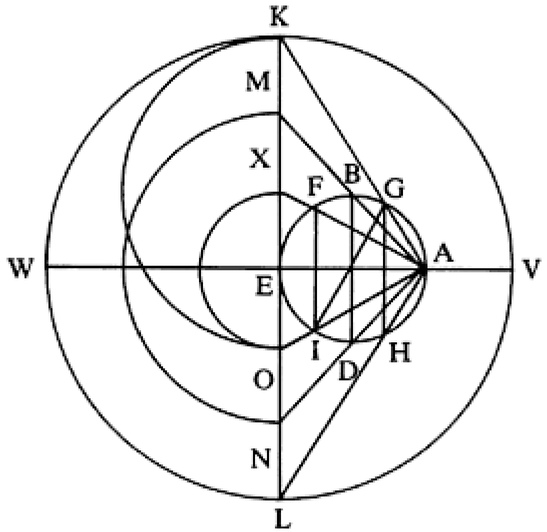


Figure 14.21

(*Tasṭīḥ al-asṭurlāb*). He constructed all the circles and points lying on the plate and the 'spider' of this instrument in a purely geometric way. Figure 14.21 reproduces al-Maghribī's drawing on which he superposed the vertical great circle of the celestial sphere ABED and its stereographic projection from point A on the horizontal plane, which was also the plane of the astrolabe, tangential to the sphere at point E. The diameter BD and the chords GH and FI parallel to it are the projections of the celestial equator and the tropics of Capricorn and Cancer, respectively, and the diameter GI is the projection of the ecliptic. The author's drawing shows quite clearly the construction of the circles with diameters MN, KL, XO and KO being the projections of mentioned circles on the astrolabe's plane.

Most interesting in this drawing is the superposition of the projections on two mutually perpendicular planes. At the end of the eighteenth century such a superposition became the basis of the 'G. Monge's method' of the modern descriptive geometry.

SPHERICAL GEOMETRY

In the first section we mentioned that in the ninth century both Theodosius's (second-first centuries BC) *Sphaerica* and Menelaus's (first century) work of the same title were translated into Arabic. Theodosius attempted to create a geometry on a sphere similar to Euclid's planimetry as introduced in the *Elements,* whereas Menelaus discovered a number of properties which geometric figures possess on the sphere and for which there was no analogy in plane geometry. These properties include the excess of the sum of the angles of spherical triangles over two right angles and the relations between the angles and the sides of such triangles. Besides this, Menelaus proved the first theorem of the spherical trigonometry, now named after him or called the theorem on the complete (spherical) quadrilateral. This term denotes a figure consisting of a spherical quadrangle both pairs of whose opposite sides are extended until they intersect, see Figure 14.22. The theorem connects the chords of the six doubled arcs of the quadrilateral. Ptolemy used Menelaus's theorem in his *Almagest* in order to solve problems in spherical astronomy. Many Arab scientists commented on and improved Theodosius's and Menelaus's *Sphaerica*. The Khwarizmian scholar Abū Naṣr ibn ʿIrāq (d. 1036), the teacher of al-Bīrūnī, accomplished an especially important revision of Menelaus's *Sphaerica*. A number of works were devoted to Menelaus's theorem. Arab scientists engaged in the study of the complete quadrilateral. They called Menelaus's theorem the proposition on the secants *(shakl al-qaṭṭāʿ)* and the complete quadrilateral itself was the 'figure of secants' in their terminology. Among the appropriate works we shall mention Thābit ibn Qurra's *Treatise on the Figure of Secants (Risāla fī shakl al-qaṭṭāʿ)*, Ḥusām al-Dīn al-Sālār's lost treatise referred to by al-Ṭūsī and Naṣīr al-Dīn al-Ṭūsī's *Removing the Veil from the Mysteries of the Figure of Secants (Kashf al-qināʿ ʿan asrār al-shakl al-qaṭṭāʿ),* also called *The Book on the Figure of Secants (Kitāb al-shakl al-qaṭṭāʿ),* and, again, in the European literature, *The Treatise on the Complete Quadrilateral*.

A large number of writings were devoted to geometric constructions on the sphere. In his *Construction of Direction on the Sphere (ʿAmal al-samt ʿalā al-kura)* Yaʿqūb al-Kindī explained the construction of a point given according to two other points on the

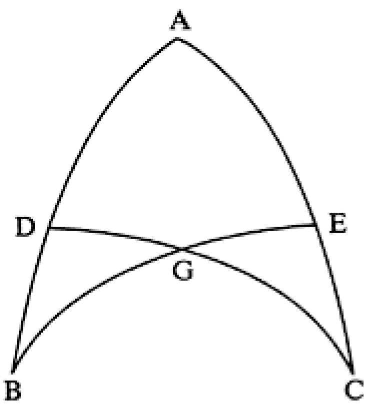


Figure 14.22

same sphere and their distances to the first one. The construction is made by compasses: circles are described from the given points taken as centres with radii equal to the given distances. In modern geodesy, this construction would have been called construction by *linear intersection*.

Al-Kindī used this construction to determine the positions of the Sun on the celestial sphere by the Sun's altitude and declination. (The complements of these two quantities to 90° are equal to the spherical distances from the Sun to the points Z and P, the zenith and the pole of the universe.) In his terminology, the 'direction on the sphere' was the direction of its radius ending at the constructed point of the sphere.

Al-Fārābī and Abū al-Wafāʾ also studied constructions on a sphere, devoting to this subject a few of the last chapters of their geometric works mentioned above. The former subdivided the sphere into regular spherical polygons with vertices coinciding with the apexes of inscribed regular polyhedra and of a certain type of an inscribed semi-regular polyhedron. The latter additionally included similar subdivisions for other semi-regular polyhedra. Ibn al-Haytham expressly devoted his *Reasoning on the Compasses for Great Circles (Qawl fī birkār al-dawā'ir al-'iẓām)* to geometric constructions on spheres.

The application of geometric methods to solving problems of spherical trigonometry played an important role in this discipline. Indeed, recall from the previous section al-Bīrūnī's and al-Kharaqī's studies in determining the azimuth of the *qibla* by stereographically projecting the celestial sphere onto the plane of the astrolabe. This determination was usually made by methods equivalent to the use of the spherical law of cosines.

Al-Khwārizmī introduced another geometric solution of the problems of spherical trigonometry. He described this in his work *The Geometric Construction of any Ortive Amplitude for [any] Zodiacal Sign and at any Latitude ('Amal si'a ayy mashriq shi'ta min al-burūj fī ayy 'arḍ shi'ta bi-l-handasa)*. Al-Khwārizmī's method came to be widely known. Taking the latitude ϕ of the place of observation and knowing the Sun's declination δ at a given day he constructed the ortive amplitude, i.e. the arc θ of the circle of the horizon stretching from the point of the East to the point of sunrise at this day. Al-Khwārizmī determined θ according to the rule

$$\sin\theta = \frac{\sin\delta}{\cos\varphi}$$

Since the arc θ is the hypotenuse of the right spherical triangle EFS (Figure 14.23), the arc δ is one of its legs and the co-latitude $90°-\phi$ is the angle lying opposite to this leg, this method is tantamount to using the spherical law of sines for the triangle EFS. Al-Khwārizmī derived the arc θ geometrically: he constructed the circle ABCD with its

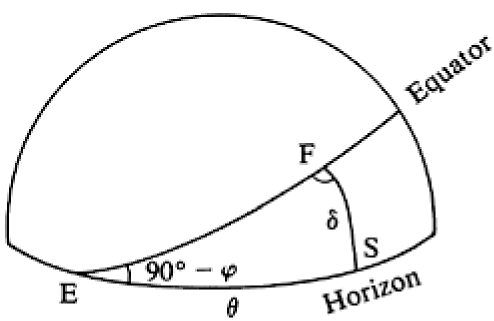


Figure 14.23

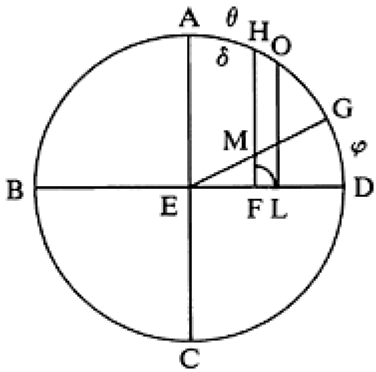


Figure 14.24

two diameters AC and BD intersecting at a right angle at the centre E of the circle; marked off the arc AH equal to δ and the arc DG equal to ϕ (Figure 14.24) on the arc AD; described the radius EC and the straight line HF parallel to the diameter AE; determined the point of their meeting M; described an arc of radius EM with its centre being at E finding the point L of its intersection with the diameter BD; and, finally, described the straight line LO parallel to AE and HF meeting the arc AD so that the arc AO was equal to the unknown ortive amplitude.

Muḥammad al-Māhānī (d. 880), the junior contemporary of al-Khwārizmī, gave a similar geometric construction of an arc equal to the Sun's azimuth *A* through its altitude *h,* the ortive amplitude θ and latitude ϕ of the place of observation. He described this in his *Treatise on Determining the Azimuth at any Hour and in any Place (Maqāla fī* ***maʿrifa*** *al-samt li ayy* ***sāʿa*** *aradta wa fī ayy* ***mawḍiʿ*** *aradta).* Al-Māhānī's construction corresponded to al-Khwārizmī's rule introduced in the latter's *Determination of the Azimuth by the [Sun's] Altitude (****Maʿrifa*** *samt min qabl* ***irtifāʿ****).* If θ is derived through δ and ϕ according to al-Khwārizmī's rule, the expression determining *A* from δ

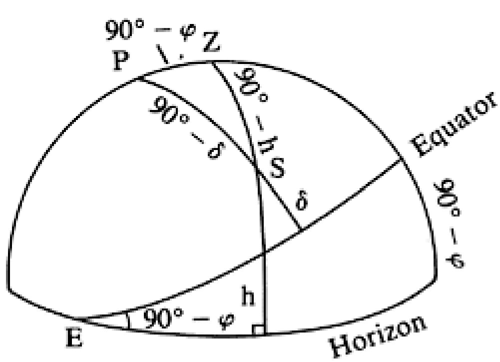


Figure 14.25

h and *ϕ* would become equivalent to the spherical law of cosines for the spherical triangle SPZ (Figure 14.25). Many subsequent Arabic *zījes* and other astronomical writings used al-Khwārizmī's and al-Māhānī's constructions.

CO-ORDINATES

In essence, while duplicating the cube by determining the intersection of two parabolas, Menaechmus (fourth century BC) was the first to use the rectangular co-ordinates regarded as straight line segments. Similar co-ordinates appeared in Euclid's lost *Conics*. The author used them in representing and studying the properties of ellipses and hyperbolas. Archimedes applied such co-ordinates in his works *On the Quadrature of the Parabola* and *On Spheroids and Conoids*. Apollonius, in his *Conics,* used both rectangular and oblique co-ordinates whereas Archimedes introduced polar co-ordinates in his treatise *On Spirals*.

However, these facts do not mean that antique scholars mastered the co-ordinate method due to mathematicians of the late seventeenth century. In ancient times the co-ordinates were closely connected with the curves which they described. In the works of Menaechmus and Euclid the rectangular co-ordinates were a segment of one of the axes of a conic section and a second segment parallel to its other axis (Figure 14.26). A segment of one of the diameters of a conic section and a segment of a chord conjugate to this diameter (Figure 14.27) mostly served as oblique co-ordinates in Apollonius's *Conics*. Finally, Archimedes' polar co-oordinates were a segment with a fixed origin and an angle between the fixed axis and this segment. If the angle remained proportional to the length of the segment, the other endpoint of the segment described an 'Archimedes' spiral'.

Thus, ancient scientists had no idea of the geometric image of equations between the two co-ordinates. They only discussed specific connections of such kind between the

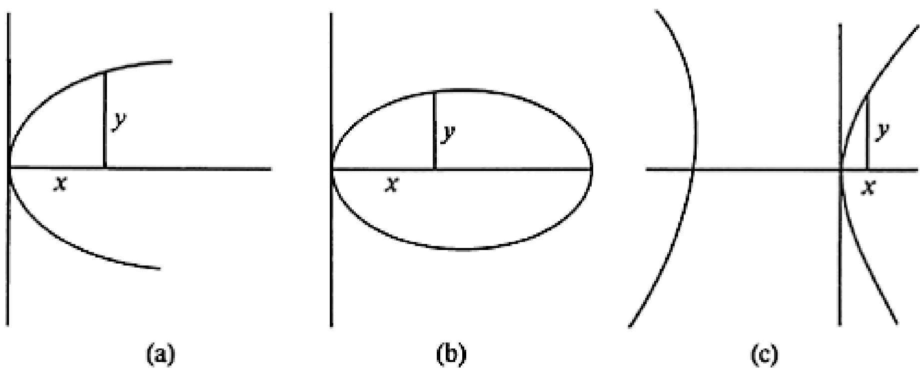


Figure 14.26

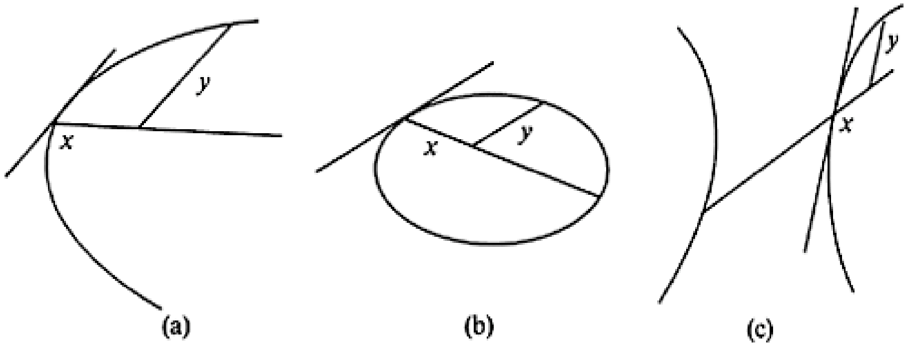


Figure 14.27

co-ordinates of points lying on curves, and they even had a special term for these connections, calling them the symptoms of the respective curves. Nevertheless, the co-ordinates as understood by Descartes and Fermat were linked with those of the ancient scholars since the modern terms abscissa and ordinate were abbreviated Latin translations of the corresponding words 'cut off from the vertex' and 'put in order' used by Apollonius.

Antique geographers used a co-ordinate system lying on the Earth's surface which at first they supposed to be a rectangle, then a sphere. The terms longitude (length) and latitude (width) had appeared at the time when the first model was in use, but they persisted even in the spherical model.

Whereas ancient mathematicians represented co-ordinates on a plane by segments and angles, considering them positive, geographers had to indicate whether latitudes on the sphere were to the north or south from the equator, a fact which corresponded to distinguishing between positive and negative co-ordinates. Note, however, that the operation of multiplication was never done on latitudes.

As a rule, the geographical co-ordinates were expressed in degrees and minutes. Antique astronomers used several types of spherical co-ordinates on the celestial sphere. These were similar to the geographical co-ordinates on the Earth's surface. Two systems of co-ordinates were fixed: the horizontal system with the circle of the horizon used as the equator and the points of the zenith and the nadir as the poles (Figure 14.28(a)); and the equatorial system with the celestial equator and the poles of the universe respectively (Figure 14.28(b)). Two other systems rotated diurnally together with the fixed stars: the moving equatorial system (Figure 14.29(a)) and the ecliptic system with the ecliptic as the equator and its poles (Figure 14.29(b)).

Algebraists (see the chapter on algebra) systematically used Apollonius's co-ordinates constructing the positive roots of algebraic equations of the third and the fourth degree by studying the intersection of conic sections.

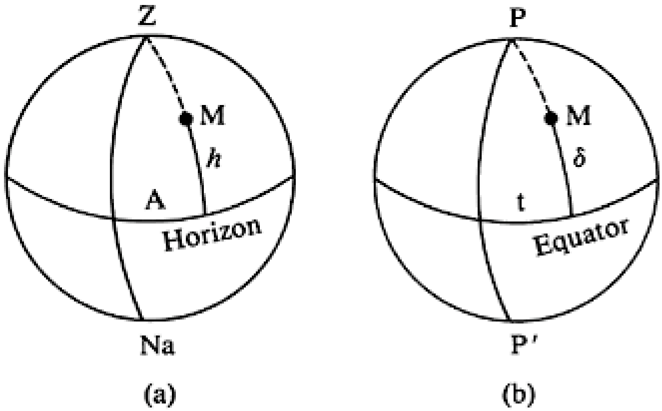


Figure 14.28

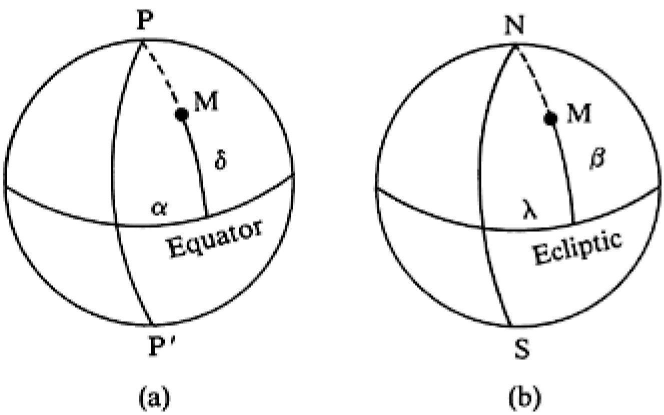


Figure 14.29

Arab scholars were well acquainted with the Arabic translations of Ptolemy's *Almagest* and with various revised forms of his *Geographia,* again in Arabic. The first such revision was al-Khwārizmī's *Book on the Picture of the Earth* ***(Kitāb ṣūrat al-arḍ).*** Therefore, scientists from the Arabic countries always used the geographical latitude and longitude as well as various co-ordinates on the celestial sphere. The term for one of the co-ordinates in the horizontal system, *samt* (azimuth), came to be also used for indicating directions on the Earth's surface.

Thābit ibn Qurra, in his *Book on Horary Instruments Called Sundials (Kitāb fī ālāt* ***al-sā'āt*** *allatī tusammā rukhāmāt),* described the position of the end of the sundial's shadow in the plane of this device by the length of the shadow (call it *l*) and its azimuth *(A)*. We may regard these parameters as the polar co-ordinates of a point on a plane. In addition the author introduced 'parts of the length' *(x)* and 'parts of the width' *(y),* i.e. rectangular co-ordinates of the same point, and indicated the rules for passing from *l* and *A* to *x* and *y* (Figure 14.30). In our standard notation these rules are

$x = l \sin A \qquad y = l \cos A$

Since in Arabic the words for longitude and latitude are expressed just as length and width respectively (*ṭūl* and *'arḍ*), and since the word 'part' was often used in the sense of 'degree', it follows that the expressions 'parts of the length' and 'parts of the width' (*ajzā' al-ṭūl and ajzā' al-'arḍ*) meant the same as 'degrees of longitude' and 'degrees of latitude'. This fact testifies that Thābit ibn Qurra took his terms for the rectangular co-ordinates from geographers.

The same problems concerning sundials that had led this scientist to ascertain the connection between rectangular and polar co-ordinates also brought al-Bīrūnī to co-ordinates in space. In *The Exhaustive Treatise on Shadows,* while considering the shadows of the gnomon projected on the plane of the horizon by the sources of light situated on the celestial sphere, al-Bīrūnī remarked that the changes in the shadows on the

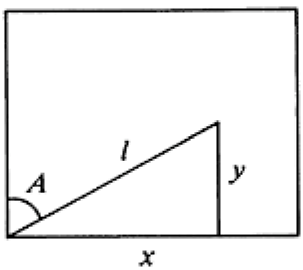


Figure 14.30

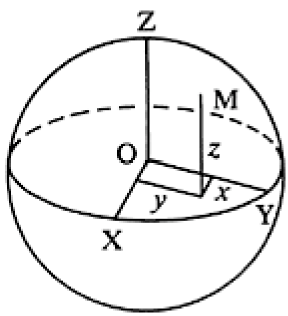


Figure 14.31

plane were occasioned by changes in the positions of the sources of light 'parallel to the diameter…of height and depth' and 'parallel to two other diameters—of length and width' (al-Bīrūnī, *Ifrād al-maqāl,* vol. 1, p. 228). The diameters of length and width are our axes OX and OY and the first diameter is our axis OZ (Figure 14.31). Thus, in determining the position of a source of light in space by the position of the 'diameters' al-Bīrūnī actually introduced spatial rectangular co-ordinates.

THE GENERALIZATION OF THE GEOMETRIC VERSION OF ALGEBRAIC IDENTITIES

Ancient Greeks used only the planimetric geometric version of algebraic identities. In the second book of his *Elements* Euclid offered a geometric interpretation of the identity

$$(a+b)^2=a^2+2ab+b^2 \tag{1}$$

(Figure 14.32) and of other quadratic identities. Archimedes, in his *Lemmas,* gave another geometric interpretation of identity (1). He proved that the complement of the semicircles with diameters *a* and *b* to the semi-circle having diameter $a+b$ (Figure 14.33) (this complement is called *arbelon*) is equal to a circle with diameter $\sqrt{(ab)}$.

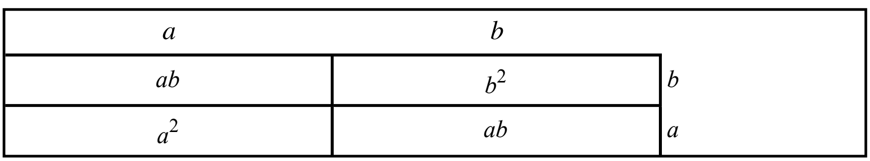


Figure 14.32

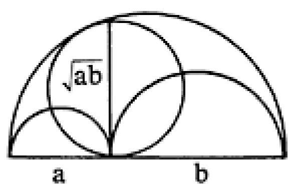


Figure 14.33

Abū Sa'īd al-Sijzī (*c.* 950–*c.* 1025), in his *Book on Measuring Spheres by Spheres (Kitāb fī misāḥat al-ukar bi-l-ukar),* generalized Euclid's and Archimedes' plane geometric version by considering problems in space. He offered a stereometric interpretation of the identity

$$(a+b)^3=a^3+3ab(a+b)+b^3$$

by dividing a cube into two cubes and three parallelepipeds. He also interpreted the identity

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

by dividing a cube into two cubes and six parallelepipeds and, again, by considering a solid obtained while rotating an *arbelon* about its diameter $a+b$ (Figure 14.34).

Two propositions placed at the end of this work testify that al-Sijzī attempted to make the following step as well. In one of these he considered a 'sphere' of diameter $a+b$ and another 'sphere' having diameter a tangential to the first one from the inside and assuming that $(a+b)^2 = 5a^2$. He stated that the first 'sphere' was twenty-five times larger

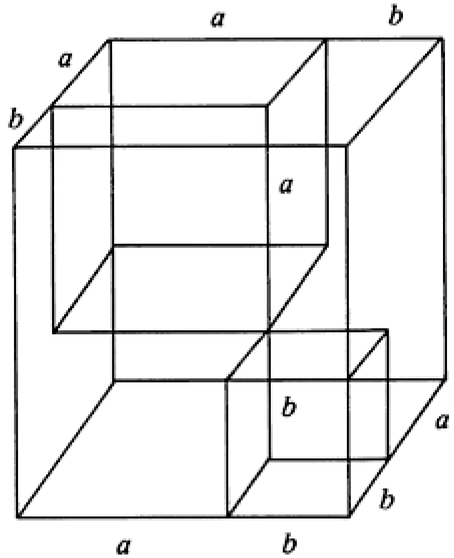


Figure 14.34

than the second one. In the usual case the ratio is $5\sqrt{5}$ rather than the second one. In the usual than 25; al-Sijzī's ratio, however, is valid for hyperspheres in four-dimensional space and it is for this reason that we have put the word 'sphere' in quotation marks.

The author said nothing about this space, and he had no appropriate terminology, but the very existence of his proposition means that he apparently thought of generalizing the theorems of the three-dimensional geometry onto the multi-dimensional case.

In Europe, the idea of multi-dimensional cubes and, for that matter, of cubes of any number of dimensions, was directly formulated for the first time in the sixteenth century in M.Stifel's commentaries on Chr. Rudolff's *Algebra (Coss)*. Rudolff studied the so-called 'Christoff's cube', or, actually, al-Sijzī's division of a cube into two cubes and six parallelepipeds.

We note a particular remark contained in the geometric works of al-Fārābī and **Abū al-Wafā'**. In the section 'Geometric constructions' we mentioned their method of constructing a square equal to the sum of three equal squares with the side of the unknown square being equal to the diagonal of a cube constructed on the given square. After delivering his description of the method, al-Fārābī remarked that 'The same takes place if we want to construct a square consisting of more than three or less than three squares' (al-Fārābī, *Al-Rasā'il,* p. 200) (a similar phrase is to be found in Abū al-Wafā's work). Evidently, these words might be interpreted as a hint at an analogous construction by means of a multi-dimensional cube. The 'hypergeometric' terms denoting algebraic degrees higher than the third one, such as *māl al-māl* (quadrato-square) for x^4, *ka'b al-māl* (cubo-square) for x^5, *ka'b al-ka'b* (cubo-cube) for x^6, may have prompted al-Fārābī (and, subsequently Stifel) to attempt such a generalization. It is also possible that al-Fārābī's lost *Book on the Introduction to the Imaginary Geometry (Kitāb al-madkhal ilā al-handasa al-wahmiyya)* was devoted to the same subject.

CONCLUSIONS

Arabic geometry, as well as Arabic arithmetic and algebra, greatly influenced the development of mathematics in Western Europe. One of the first West European geometric works was Abraham bar Ḥiyya's (*c.* 1070–*c.* 1136) *Book of Measurements (Liber embadorum)*. In the Latin literature the author was called Savasorda, a word derived from the Arabic *ṣāḥib al-shurṭa* (commander of the guard). He wrote his book in Hebrew and later Plato of Tivoli translated it into Latin. It contained many rules of Arabic geometric calculations which sometimes involved algebra.

In the mid-twelfth century, Savasorda together with Plato of Tivoli translated Arabic works, e.g. many books written by al-Khwārizmī, Thābit ibn Qurra and Ibn al-Haytham, into Latin.

Leonardo Pisano (*c.* 1170–*c.* 1250) wrote his *Practica Geometriae* while being under a considerable Arabic influence. The book contained a number of planimetric and stereometric theorems complete with proofs. In his arithmetical and algebraic writing *Liber Abaci* the same author used terms of Arabic origin; e.g. *'figura chata'* originated from the Arabic *shakl al-qaṭṭā'* (the theorem on secants).

In Europe, just as in the Arabic East, the stereographic projection (see the section 'Geometric transformations') was extremely popular. By means of this projection European instrument makers made astrolabes in the Arabic manner. That Europeans followed the Arabs is evident since the names of the stars engraved on the 'spiders' of European astrolabes were mostly transcriptions, often corrupted, of the corresponding Arabic names. No wonder that in many instances the modern European names of stars are corrupted versions of their Arabic names.

Witelo, a Polish scientist of the thirteenth century, wrote his *Perspectiva* (whose supplement was Kepler's celebrated *Astronomia pars Optica*) being strongly influenced by Ibn al-Haytham's *Book of Optics*.

In the sections 'The theory of parallel lines' and 'Geometric transformations' we mentioned Alfonso's treatise *Straightening the Curved* and Levi ben Gerson's commentaries on Euclid's *Elements,* both of them written in the fourteenth century in Hebrew.

In the fifteenth century, after the Turks had conquered Constantinople, many Byzantine Greeks fled to Western Europe bringing Arabic manuscripts with them. Thus, two manuscript copies of Pseudo-Ṭūsī's *Exposition of Euclid* found their way to Italy, and the work itself was published in Rome from one of these copies. We mentioned this fact in the sections 'The foundations of geometry' and 'The theory of parallel lines', indicating that the proof of Euclid's postulate V contained in this book had influenced Wallis's and Saccheri's theories of parallel lines.

For all that, European mathematicians took in the information about the Arabic geometric literature that did come to Europe in different ways: through Spain in the twelfth century; owing to the Mediterranean trade during the thirteenth-fourteenth centuries; and with the Byzantine Greeks in the fifteenth century. This fact played an important role in the origin and development of geometry in Europe.

However, Europeans—as far as we know for the moment—remained ignorant of some discoveries made by Arab scholars and subsequently found them out anew. Far from all the works of al-Khwārizmī, Thābit ibn Qurra and Ibn al-Haytham were translated into Latin and medieval Europe knew nothing about the works of al-Bīrūnī. Neither were the European scientists acquainted with many of al-Fārābī's and Abū al-Wafā's geometric constructions; with affine transformations used by Thābit ibn Qurra and his grandson Ibrāhīm ibn Sinān; and with Arabic treatises on the theory of parallel lines where Euclid's postulate V was explicitly replaced by one or another equivalent propositions.

NOTES

1 With regard to the system of postulates and axioms the extant copies of the *Elements* (the earliest of these date back to the ninth century) contain somewhat variant readings. In particular, in some manuscripts the fifth postulate is called axiom XI. We follow the now generally accepted text established by J.L.Heiberg at the end of the nineteenth century.

2 In the extant manuscript copies of the *Canon Masudicus,* this arc is marked off not along the celestial equator, but on the circle of the meridian.

15
Trigonometry

MARIE-THÉRÈSE DEBARNOT

FROM GEOMETRY TO TRIGONOMETRY

As an auxiliary in the study of the movement of heavenly bodies, trigonometry, originally, goes back at least to Hipparchus, to whom the first table of chords is attributed. Around the sixth century the Indian scientists had already replaced the archaic chord of the double arc by its half, in other words *R* times our sine (labelled here Sin for *R* sin), giving the radius R of the circle or the sphere several values (150, 3438, 120 etc.). The contribution in this area of Indian science is not limited to the introduction of the sine. Nevertheless, in the eyes of the Arab astronomers of the ninth century, the *Almagest* will not take long to overtake the Indian *siddhanta*. The book seduces by the rigour of its exposure, by its demonstrations and by the observation programmes it suggests. As paradoxical as it might seem, the enormous building constructed by Claudius Ptolemy in his famous *Almagest* rests essentially on very elementary geometrical propositions. In the quite complex calculations relating to planetary models, we constantly arrive at Pythagoras's theorem and the chord that would represent the side of the right angle of a right-angled triangle of which the hypotenuse would be equal to the diameter of the circle of reference (with R=60, as common usage in sexagesimal numeration). Thus sides and angles of plane triangles are obtained from one another. The same geometric language is found in chapter 10 of the first book, applied to the construction of the table of chords, where he implicitly covers the addition formulae of the arcs. As for spherical astronomy, it is apparently reduced to a dozen simple applications of the theorem of Menelaus.

This is schematically the trigonometric structure of the *Almagest,* abstracting, for the time being, from some more subtle devices. Instructed by the Greek and Indian texts, the first Arab astronomers, in just some decades, are going to equip themselves with spherical astronomy which, although it might seem confusing in its terminology and its themes, is capable of solving any kind of problem. It is only a century and a half later, around the year 1000, that the undertaken reform will produce a mathematical formulation with the appearance of the first accounts of the spherical triangle. The notions will then be clarified, namely in relation to the tangent function, already introduced since the ninth century. At this same time, the interest in formulating a specific method and language becomes clear. In this Buyid time, when the scientific centres are numerous and active, we can truly speak of the emergence of trigonometry. From then the new science will become the object of independent treatises, whilst research of a better precision in the reading and construction of sine tables will give incentive to other works.

Throughout this chapter, we shall follow the path that has led to the birth of the particular technique that we call trigonometry. It will be necessary to refer to texts, to quote some formulae: the present state of our knowledge will not allow us to erect a complete inventory. We shall be careful not to search systematically for the precursors of Regiomontanus, Viete, Rheticus and other founders of trigonometry in Europe.

Comparisons are risky in the sense that, in the West, trigonometry was built from knowledge already compiled outside the astronomical context which had been responsible for its birth, five centuries before, in the Abbasid country. According to its use the same formulae can in effect have a different meaning, presenting more or less interest. It is a point to which we shall return with regard to the formulae of any spherical triangle and the notion of polar triangle. In the same sense, it will be improper to confuse for instance the simplification occasionally applied by Ibn Yūnus or al-Kāshī when in some rules of astronomy they replace some products of sines and cosines by sums, using the process of calculation called 'prostapheretic',[1] known in Europe in the sixteenth century and having had its utility before the introduction of logarithms.

It is frequent in mathematics that notions attractive at a certain time afterwards become obsolete. We have just seen an example. At the particular time we are interested in, it is the case with the versed sine (Vers θ=R(1−cos θ), taken by Arab authors from Indian science, which in their treatises plays in a certain sense the role of our cosine. In the absence of all orientation and sign notion, the versed sine has the advantage of taking distinct values, whether the angle is acute or obtuse. Not long ago, there was a similar interest in finding spherical trigonometry formulae in a logarithmic form; this has become pointless and the resolution of triangles has lost its place in astronomy books. Like other disciplines, trigonometry has followed the standard evolution of mathematics; it is also convenient to treat as relative the attention given to each period of its development. For us, the Arabic period is that of the first formulae for a triangle, the first definitions and the introduction of the tangent function. We shall forget the subject that was undertaken later within analysis—circular functions—to return to the period when trigonometry takes shape and is distinguished from geometry.

THE SPHERICAL CALCULATION OF THE *ZĪJ*

The double legacy of Arabic spherical astronomy, the problems that make it richer and the ways chosen to solve them in the ninth century, will be established as soon as a mathematical tool aiming at an easier approach is forged. It is also important to know its components, even if this means getting slightly away from the subject of our study.

A first constitutive element of spherical calculation, such as appears abundantly developed in the *zīj* (astronomical tables), comes from its Greek heritage. It holds to the primordial role of the ecliptic, the reference circle of the planetary movements. From there begins a technique of decomposition of problems which reduces, in advance, the number of useful formulae. As in the *Almagest* everything is, so to speak, brought back to the ecliptic: the angles with the verticals (for the parallax) and with the horizon (visibility), the points—or 'degrees'—of the ecliptic associated with all heavenly bodies ('degree', 'degree of passage' at the meridian, 'degrees of rising and setting'), the points situated at a given moment at the meridian or on the horizon (from which the ascendant, or horoscope, of the astrologers) which fix the position of the sphere dragged along in the diurnal movement. An important notion is that of the oblique ascension[2] for which the table, which must be calculated at each latitude, is looked up in the *Almagest* in order to obtain the longitude of the ascendant. Thus it is only necessary to apply the theorem of Menelaus to simple problems, often from a quadrilateral formed by quarters of great circles.

It is known that proposition III, 1 in Menelaus's *Spherica* establishes a relation between six arcs carried by three of the great circles supporting the sides of a complete quadrilateral; when the sides are quadrants the relation is equivalent to a formula of the right-angled triangle.[3] Whoever was instructed in the astronomy of the *zīj* would know for instance that 'the sine of the declination of the sun—or of a 'degree'—is obtained on dividing by the sphere's radius, the product of the sine of the sun's longitude and the sine of its greatest declination (the obliquity of the ecliptic)'. The rule that links the hypotenuse, a side of the right-angle and the opposite angle of a spherical right-angled triangle is a result of the application of Menelaus's theorem to the quadrilateral of which the contours are drawn once the problem is formulated (see Figure 15.1).[4] It is the same kind of calculations that we find in the *Almagest* with the difference that, in Ptolemy's

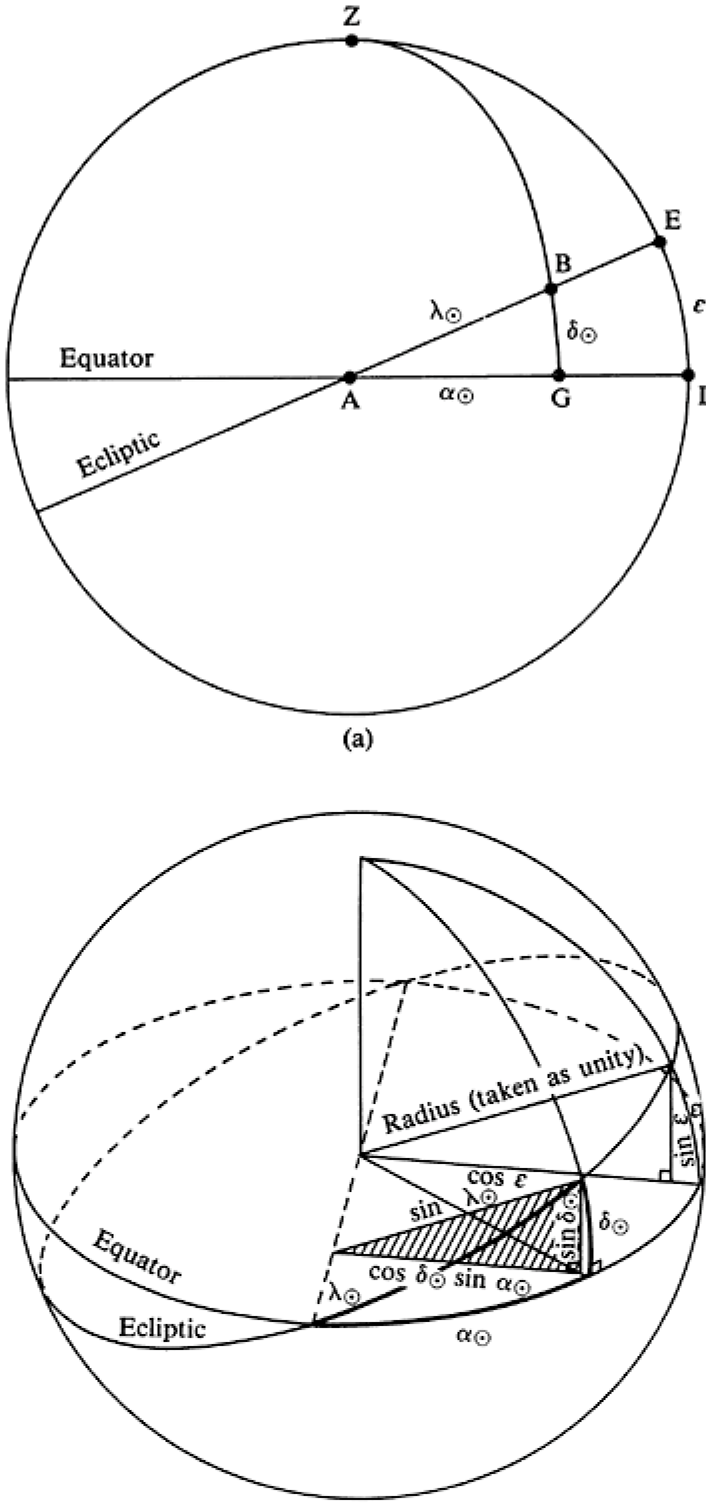

Z
E
B
λ⊙
δ⊙
ε
Equator
A
α⊙
G
D
Ecliptic
(a)
Radius (taken as unity)
cos ε
sin λ⊙
sin δ⊙
sin ε
δ⊙
Equator
cos δ⊙ sin α⊙
λ⊙
Ecliptic
α⊙

Figure 15.1

book, from the letters of a figure, we learn how to calculate the chord of twice a given arc, knowing that a certain relation between chords is formed of two others. We devise that, even without any symbolism, the formulation of rules represents an important step towards finding analogies and towards the idea of common mathematical expressions.

Rules like the one giving the sun's declination can be found in texts from India, like the *Khaṇḍakhādyaka* from Brahmagupta, which was known before the *Almagest* and the *Handy Tables*. Its context is very different. Here, no demonstrations, no figures, no representations on the surface of a sphere, but statements in verse expressing the similarity of plane right-angled triangles of which the sides are sines, versed sines, the shadow of the gnomon, the radius of a sphere or the sums of these quantities. In the case cited, the two triangles, which must be represented on two parallel planes inside the sphere, have as hypotenuse the sine of the sun's longitude and the radius of the sphere respectively; as homologous sides, they have the sine of the sun's declination and the sine of its largest declination.[5] Without any other means at its disposal, the spherical astronomy of the *siddhanta* is more basic than that of the *Almagest*. However, it supplies other rules, like the equivalent of $\sin \alpha_{\odot} = \sin \lambda_{\odot} (\cos \varepsilon / \cos \delta_{\odot})$ (Figure 15.1(b)) which, linking the arcs of four circles, can not only result from a single application of Menelaus's theorem. It mainly develops the general notion of azimuth and the important idea of linking through a formula the measurement of time with the height of a heavenly body of given declination. The Indian method of plane triangles succeeds in fact where we would call for the cosine formula: calculation of the hourly angle as a function of height, relation between the azimuth and the height.[6]

Enriched with the two learned methods, the pioneers of astronomy in the ninth century do not confine themselves to making a synthesis which gives statements the form of clear instructions expressed with the help of the Indian sine and versed sine with R=60, the value in the *Almagest*. A careful reading of Ptolemy's book leads to a refinement of the methodology. This is true of the spherical calculation from which some approximations by plane triangles are suppressed (parallax, visibility, eclipses).[7] They break free from the constraint represented by the table of oblique ascensions; in the *zīj,* it is the problem of the 'ascendant without table', which does not necessarily have an astrological connotation. Thanks to the azimuth, which is measured on the 'Indian circle' and becomes a common notion with the *qibla,* the position of the heavenly bodies and their local coordinates are linked from now on: the classical calculation of 'azimuth ascensions' is the same as the determination of the hourly angle knowing the azimuth. The ecliptic co-ordinates are calculated from the declination and from the 'degree of passage'; in the reports of the *Almagest,* they were approximately evaluated with the help of known positions of nearby heavenly bodies. Another subject which in its own right creates an abundant literature, the *qibla,* concerns purely astronomical questions; it is a calculation that relates to a changing of co-ordinates (azimuth, knowing the hourly co-ordinates) and that is seen as such when it goes through the determination of the height of the zenith of Mecca to the place considered. Many other subjects are tackled in the *zīj*. We shall stop this incursion on the rather technical area of spherical astronomy here. When the historians of science mention its development in the Arabic period, they readily quote the *qibla:* this accounts poorly for the complexity of the calculation of the *zīj,* owing to its heterogeneous composition and the prodigious expansion of astronomy in the ninth century. As for astrology, it will only acquire its spherical techniques after the simplifications brought by the formulae of the triangle.

How are we to solve the new problems, some of which arise from the resolution of any triangle? There are some fruitless efforts; we know of them in particular through critical works. The authors of tables will nevertheless rapidly become rivals in the contribution of varied solutions. The most remarkable idea is certainly the application of auxiliary functions; we shall return there with regard to the tangent. To the different geometrical methods is added the beautiful graphical procedure known as the 'analemma', which is in the nature of our descriptive geometry.[8] Alī this leads to rules for the calculation of unknown arcs. We only want to retain here those that proceed from reasoning 'at the surface of the sphere', as in the *Almagest*. The expedient found to get round the difficulties consists then, naturally enough, in passing from circle to circle until we obtain the arc we want. This is a technique whose singularity escaped the medieval Arab commentators as it was so familiar to them. A whole terminology applied to auxiliary arcs starts being applied as common practice; it traces, we might say, the paths of the most common methods. Thus, until the end of the tenth century, all kinds of rules are going to be accumulated in the *zīj,* whose steps, susceptible in most cases to a demonstration by Menelaus's theorem, almost always recover the same formulae of the spherical right-angled triangle.

TOWARDS THE FORMULAE OF A TRIANGLE

The introduction of the tangent function in the ninth century will go almost unnoticeably. The discovery of theorems substituting for the quadrilateral is in contrast an outstanding event followed by arguments about priority. For the contemporaries of this renewal of the techniques of astronomy, Menelaus's theorem is without contest the only spherical formula applied by their predecessors. During the first two centuries, mathematical research seems in fact to crystallize around this theorem, although some rules in the *zīj* had perhaps already been obtained differently, read in some way at the surface of the sphere. At the same time, the astronomers began to break free from Menelaus's theorem by directly establishing usual formulae.

We need to remember that numerous astronomy texts from the ninth and tenth centuries are without demonstrations. When the opportunity offers they will be the object of subsequent studies. We know for instance that al-Bīrūnī wrote two enormous commentaries—both lost—on the tables of al-Khwārizmī and of Ḥabash al-Ḥāsib. It is clear, and we have seen with regard to the declination of the sun, that the same result can often be obtained in many different ways. It is on the whole the context which suggests the author's mode of demonstration. Thus, it has been established that Ibn Yūnus, following his predecessors al-Battānī and Ḥabash, uses methods 'inside the sphere'[9] in his *Great Hakemite Table,* as the numerous variants proposed in the solution of each problem are read on the same analemma. When a formula is repeatedly applied to the same simple spherical configuration, we can ask ourselves whether the author returns each time to a direct demonstration or to a theorem that is as difficult to use as the theorem of Menelaus, or simply whether he does not transpose the rule obtained at first. Luckey has made the same remark on the subject of the statements of Thābit ibn Qurra about the sundial. The question is even clearer with regard to the whole of the spherical calculation of the *zīj* of Ḥabash. The stages of reasoning 'at the surface of the sphere', going through the calculation of auxiliary arcs, correspond in effect to four elementary rules, stated at the beginning, of which one, the Indian formula already mentioned for the right ascension of the sun, is not immediate through Menelaus's theorem. If this was the process used, it explains the facility with which the difficult conversions of local co-ordinates into equatorial or ecliptical co-ordinates were solved in this treatise.

Be that as it may, Ḥabash does not state any formulae for the triangle. We shall return to the important contribution of this ninth-century astronomer. The great Thābit, whose activity extended to all the fields of mathematics and astronomy, was one of the numerous authors to be interested in Menelaus's theorem. Known from this epoch in the exposition of the *Spherica,* the demonstration of the theorem occupies all Chapter 13 of the first book of the *Almagest.* 'Later developments concerning this expression and the different cases to be contemplated were introduced by Abū al-ʿAbbās al-Faḍl ibn Ḥātim al-Nayrīzī and Abū Jaʿfar Muḥammad ibn al-Ḥusayn al-Khāzin in their respective commentaries on the *Almagest*', says al-Bīrūnī. 'Abū al-Ḥasan Thābit ibn Qurra dedicates a book to the compound ratio, to its different types and to its applications, and another one to the quadrilateral *(al-shakl al-qaṭṭāʿ),* making this theorem easier to use. Numerous modern authors'—still according to al-Bīrūnī—'have gone deeply into this question, such as Ibn al-Baghdādī, Sulaymān ibn ʿIṣma, Abū Saʿīd Aḥmad ibn Muḥammad ibn ʿAbd al-Jalīl al-Sijzī and many others. They give it particular interest because it was in some way the cornerstone of astronomy; without it none of the calculations mentioned earlier (the calculations of the *zīj*) would have been possible.'

The only spherical formula of the prestigious *Almagest,* proposition III, 1 of the *Spherica,* lends itself to mathematical studies in cases where it is possible to use a compound ratio, in the absence of all symbolism (Figures 15.2 and 15.3).[10] As in the *Spherica* and the *Almagest,* but without chords, the theorem is stated and demonstrated in two forms each expressing that a ratio of sines is composed of two others. In the form called *tafṣīl* (diaereris) it refers to (Figure 15.3) $\mathrm{Sin}\,\widehat{AE}/\mathrm{Sin}\,\widehat{EB}$ or $\mathrm{Sin}\,\widehat{GD}/\mathrm{Sin}\,\widehat{DB}$; in the other case (tarkīb, synthesis) it refers to $\mathrm{Sin}\,\widehat{AB}/\mathrm{Sin}\,\widehat{BE}$ or $\mathrm{Sin}\,\widehat{GB}/\mathrm{Sin}\,\widehat{BD}$[11] The Arab authors make other distinctions according to the arc required. Thābit considers eighteen cases, after a very elegant

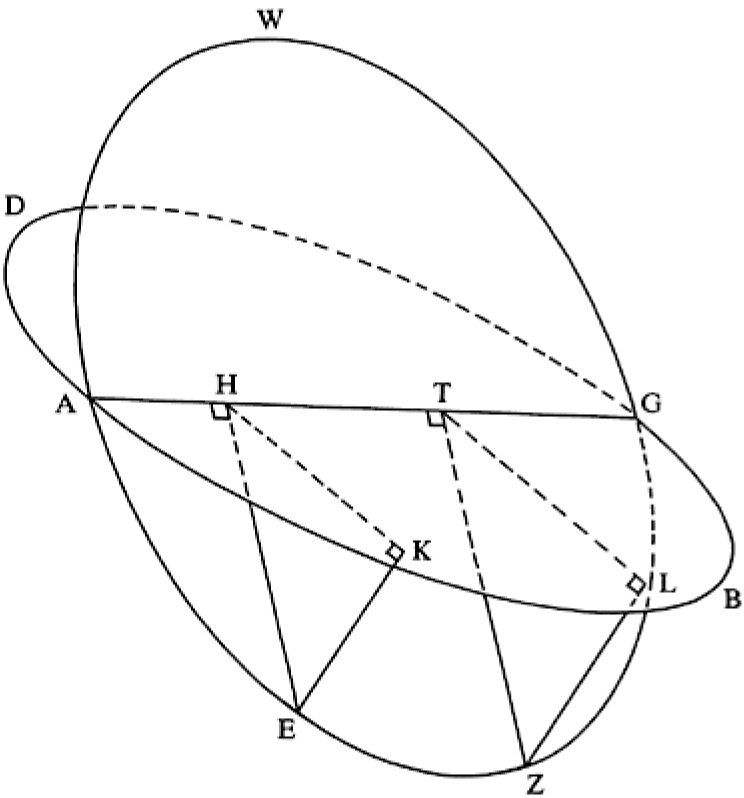


Figure 15.2

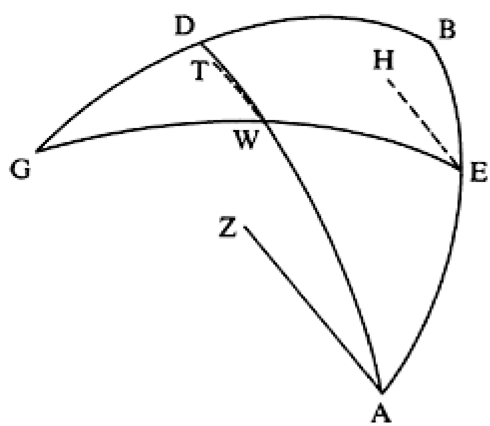


Figure 15.3

demonstration which, instead of appealing to the plane theorem, reduces the spherical theorem to the identity

$$\frac{a}{b}=\frac{a}{c}\cdot\frac{c}{b}$$

which translates it in projection to a straight line.[12] Such studies, we see, while emphasizing the difficult aspect of the theorem, support the mode of reasoning 'at the surface of the sphere' in astronomy, and constitute a first step to the elaboration of a specific mathematical technique.

Abū al-ʿAbbās al-Nayrīzī, one of the authors quoted by al-Bīrūnī, has a method for the *qibla* by Menelaus's theorem. We have few texts like this one, including original calculations explicitly carried out with the help of the quadrilateral, but those of Abū Naṣr ibn ʿIrāq and Abū al-Wafāʾ al-Būzjānī, the two main craftsmen, with Abū Maḥmūd al-Khujandī, about the revival that took place at the end of the tenth century are still extant. That is, there was no intermediate mathematical statement to precede what is conventionally known, more or less correctly, as the discovery of the general theorem of sines.[13] The possibility of a common source has been put forward to explain the coincidence in the contributions of the three astronomers from Khwārizm, from Baghdad and from Rayy. This hypothesis is contrary to al-Bīrūnī's statement in his *Keys of Astronomy,*[14] dedicated to the introduction of new theorems. In reality, the similarity of the statements has no other origin than the contents of astronomical texts, and it is probably not by chance that the three systems of formulae destined to replace Menelaus's theorem are proposed in the frame of the consistent astronomical studies.

The name of Abū Maḥmūd al-Khujandī (died around 1000) remains attached to the famous sextant *fakhrī,* more than twenty cubits high, graduated in minutes of arc, which was constructed at Rayy—close to the present Teheran—under the patronage of the rich Buyid sultan Fakhr al-Dawlā. Al-Bīrūnī described the beautiful instrument which he had the opportunity to examine in the company of Abū Maḥmūd. In his *Keys* he echoes discussions which were taking place amongst the small scientific community of Rayy about a theorem over which al-Khujandī disputed priority with Abū al-Wafāʾ and which he named 'the canon of astronomy'. It concerns the formula that we know as the 'rule of the four quantities'.[15] Al-Khujandī presents to al-Bīrūnī a treatise on the observation of the heavenly bodies, at the start of which he had established this theorem applying it afterwards throughout the book. At Rayy there is also another astronomer, Kūshyār ibn Labbān, who has taken up in one of his works the exposition of Abū Maḥmūd, reshaping it and adding to the theorem the word *al-shakl al-mughnī*[16] which will stay attached to it. As al-Bīrūnī notes, the long demonstration of al-Khujandī is quite different from those of Abū al-Wafāʾ; it mentions, however, the similar right-angled figures by which, 'in a much easier way', Abū al-ʿAbbās al-Nayrīzī (died around 922) and Abū Jaʿfar al-Khāzin (died around 961–71) have already

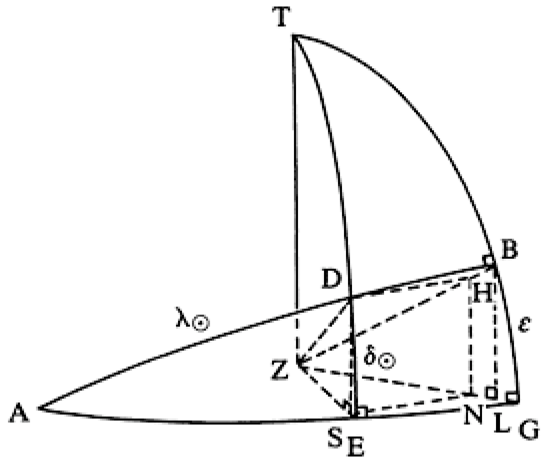
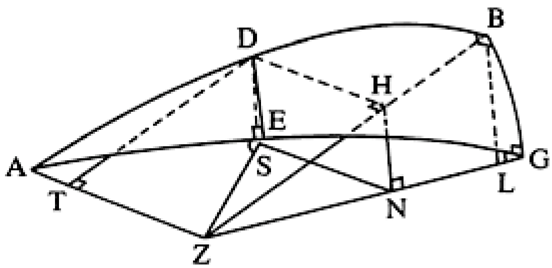


Figure 15.4



Figue 15.5

found the rules of the *Almagest* (Figures 15.4 and 15.5).[17] By different ways the astronomy of the *zīj* leads to the same triangle formulae. Abū Maḥmūd al-Khujandī is not a mathematician of the first level. The necessary reform will be the combined work of Abū Naṣr ibn ʿIrāq and Abū al-Wafā' al-Būzjānī.

THE THEOREMS OF ABŪ NAṢR AND ABŪ AL-WAFĀ'

For al-Bīrūnī and his contemporaries, the simplification brought about in their era to the techniques of astronomy involves a 'figure' which can be demonstrated as sufficient to replace the quadrilateral. The significant term *al-shakl al-mughnī* which is applied to it recovers very well the indispensable aspect of the theorem—rule of four quantities and relation of the sine of a right-angled triangle—as well as its extension, rightly considered remarkable but of less interest, the general theorem of the sine. Another formula is retained, the rule of the tangents[18] of Abū al-Wafā', which will take the name *al-shakl al-ẓillī, al-ẓill* (the shadow) designating the tangent. At base, the approaches of Abū Naṣr and Abū al-Wafā', are quite different.

The Emir Abū Naṣr ibn 'Irāq (died around 1036) did not leave, as his renowned student Abū al-Rayḥān al-Bīrūnī (973—after 1050) did, a work covering all the areas of the sciences of his time. His writings concern astronomy, especially mathematics, and some specific aspects of geometry. It was him who gave the first complete version of the *Spherica* of Menelaus, abandoned by his predecessors in the face of some difficulties in the third book. His translation is considered as the closest to the Greek text, which is lost today. This man who came from a high social background discerned the exceptional qualities of the young Abū al-Rayḥān whom he taught in mathematical studies. Their collaboration blossomed for a long time at Khwārizm before they were exiled, with other scholars of Kāth, at the court of Maḥmūd, the powerful chief of the new Ghaznevid empire. The account of the Keys goes back to the Khwārizm period. Abū al-Wafā' al-Būzjānī (940–97 or 98) was then well known. Coming in his youth to establish himself in Baghdad where his uncles were astronomers, he dedicated his life to astronomy and mathematics. Al-Bīrūnī quotes his observations, notably that of a lunar eclipse carried out jointly in order to deduce the difference in longitude between Baghdad and Kāth. Abū al-Wafā' is also the author of various mathematical works, both theoretical and practical. The trigonometric calculation occupies an important place in his *Almagest* written at the end of his life, which was probably unfinished as the only manuscript that we have contains just the seven treatises mentioned by al-Bīrūnī.

The circumstances in which the new theorems were introduced are reported by al-Bīrūnī. Two relations of a right-angled triangle were first established by Abū Naṣr in his book of *Azimuths*. Intending to find demonstrations of the different rules collected by Abū Sa'īd al-Sijzī, Abū Naṣr mainly applies Menelaus's theorem; the two formulae are not even expressed. This application of the quadrilateral and the compound ratio is criticized by Abū al-Wafā', who studies the book in Baghdad. The methods of his *Almagest* are—he says—'more concise and better than those'. In response, Abū Naṣr dedicates to al-Bīrūnī a *Risāla*[19] in which he expounds the ideas that he was not able to develop in the book of *Azimuths*. Known by the title *Epistle on spherical arcs,* for Abū Naṣr the *Risāla* is his thesis *On spherical triangles,* a title which corresponds better with the contents. A year later, al-Bīrūnī will receive the first seven treatises of the *Almagest* from Abū al-Wafā'. Compiled soon afterwards (between 994 and 1004), the *Keys* brings up the controversy created by these various writings, testifying to the importance given to the event and also to the life that animates all these scientific centres dispersed throughout the Abbasid caliphat. But let us go back to the statements of the *Risāla* and of the *Almagest* of Abū al- Wafā', in the first chapter of the second book.

The exposition of the *Risāla* is arranged on the basis of the general theorem of sines:

> In every spherical triangle formed by arcs of great circles, the sines of the sides are proportional to the sines of the arcs measuring the angles opposite them.

Abū Naṣr establishes four distinct formulae, which he applies to the problems in the

Almagest:

• the general theorem of sines,

$$\frac{\text{Sin } a}{\text{Sin } A} = \frac{\text{Sin } b}{\text{Sin } B} = \frac{\text{Sin } g}{\text{Sin } G} \qquad (1)$$

• the corresponding relation in a right-angled triangle (at G),

$$\frac{\text{Sin } a}{\text{Sin } A} = \frac{\text{Sin } g}{R} \qquad (2)$$

• and, always for a right-angled triangle at G, two relations[20] close to the formula cos *A*=cos *a*. sin *B*,

$$\frac{\text{Cos } a}{\text{Cos } A} = \frac{\text{Sin } g}{\text{Sin } b} \qquad (3)$$

$$90^\circ - A = \delta_B(90^\circ - a) \qquad (4)$$

Only the theorem of the sine is the subject of a direct demonstration, which does not lack in elegance and includes the particular case of relation (2), demonstrated earlier in the book of *Azimuths* (Figures 15.6 and 15.7).[21] Relation (3), from the book of *Azimuths,* and formula (4) are deduced immediately from (2) in the *Risāla* by associated spherical triangles.

All the formulae of Abū Naṣr are about triangle relations. The fundamental double theorem of Abū al-Wafā', in contrast, connects the arcs formed by a pair of associated right-angled triangles:

> If two arcs of great circles intersect on the surface of a sphere and if one takes any points on one of them, the ratios one to the other of the sines of the arcs included between these points and the point of intersection are equal to the respective ratios of the sines of their first inclinations and the shadows of their second inclinations.

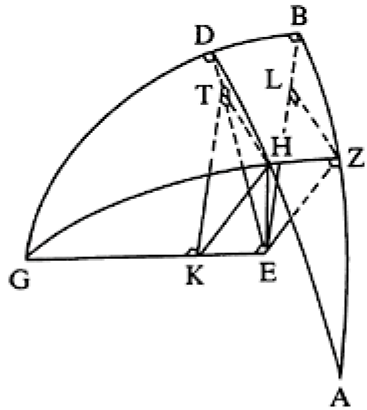


Figure 15.6

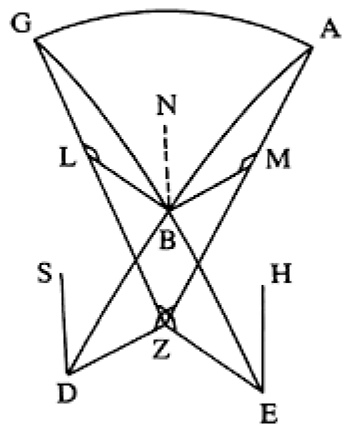


Figure 15.7

We recognize (Figure 15.8):

• the rule of four quantities,

$$\frac{\text{Sin } g}{\text{Sin } g'} = \frac{\text{Sin } a}{\text{Sin } a'} \quad (5)$$

• and the tangent rule,

$$\frac{\text{Sin } b}{\text{Sin } b'} = \frac{\text{Tan } a}{\text{Tan } a'} \quad (6)$$

As a corollary we obtain from (5):

• a relation of the right-angled triangle (at G),

$$\frac{\text{Cos } g}{\text{Cos } a} = \frac{\text{Cos } b}{R} \tag{7}$$

• and the general theorem of the sine,

$$\frac{\text{Sin } a}{\text{Sin } b} = \frac{\text{Sin } A}{\text{Sin } B} \tag{1}$$

without passing through the intermediate statement of relation (2). The two parts of the main theorem are established directly, the first in two ways, one of the methods being inspired by the demonstration in *Almagest* for Menelaus's theorem (Figure 15.9).[22]

The models elaborated by Abū Naṣr and Abū al-Wafā' have a different inspiration. From the point of view of applications they can be reduced to four theorems, *al-shakl al-mughnī* (formulae (1), (2) and (5)), *al-shakl al- ẓillī* (formula (6) which will also take the form (Sin *b*)/*R*=Tan *a*/Tan *A*) and two theorems of less interest, relation (7) and the variants (3) and (4) of the relation (Sin *B*)/*R*=Cos *A*/Cos *a*. Stemming from the rules of the *zīj,* these formulae fulfil comprehensively the necessities of astronomical calculation. Al-Bīrūnī shows this in his *Keys* by solving all the classical problems by means of a single 'figure which dispenses [from the quadrilateral]'. In the *Almagest* of

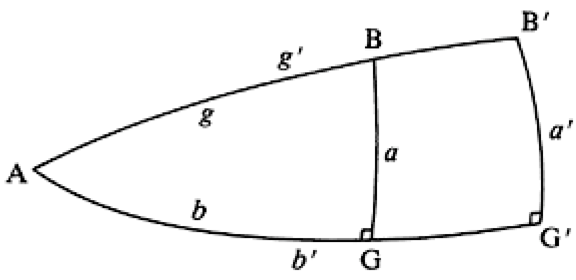


Figure 15.8

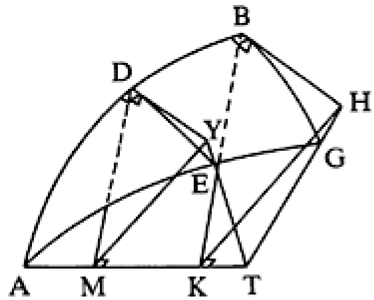


Figure 15.9

Abū al-Wafā', the spherical calculation, dealt with by a profusion of methods throughout books II to V, reveals the efficiency and flexibility of the formulae in chapter II, 1. Later, the Arab authors will express the six relations of the right-angled triangle, but their astronomical texts will only retain the tool forged by Abū al-Wafā' and Abū Naṣr. This also applies to the *zīj-i Khāqānī* of Jamshīd Ghiyāth al-Dīn al-Kāshī (died in 1429), one of the last great mathematicians and astronomers of Islam. Only the first three theorems, and mainly the rule of the four quantities, are applied there. One of the most famous astronomy texts in the Arabic West, the *Iṣlāḥ al-Majisṭī* of Jābir ibn Aflaḥ al-Ishbīlī (twelfth century) does not even mention the tangent. Translated into Latin, this text was one of the sources of *De triangulis* by Regiomontanus (1436–76). The relation (Sin *B*)/*R*=Cos *A*/Cos *a,* from the treatise of Jābir, was known in the West as Geber's theorem.[23]

THE TANGENT FUNCTION

The notion of triangle sustains all the trigonometry of Abū Naṣr. The adoption of the triangle as a basic configuration will be extended in the trigonometry treatises. With the 'shadow figure' of Abū al-Wafā', it is the tangent that enters definitively in the astronomical calculation. It will need much time for the idea, which seems so simple, of using the ratio of the sine to the sine of the complement to be identified and to break away from the neighbouring idea of the shadow of the gnomon. Despite the term 'shadow' which is applied to it, it does not seem to stem directly from it. By one of the detours of which other examples exist in the history of mathematics, the tangent appears in the track of the more sophisticated auxiliary functions, born from the analysis of spherical calculations that depend in a similar way on two parameters.

On reading again the texts which precede the introduction of the tangent, one is surprised to note the number of occasions where this function has been defined and tabulated. It is first Menelaus's theorem that requires the tangent in some of its applications. Lacking a table giving the tangent of α or its equivalent, Crd 2α/Crd(180°−2α), as a function of a, the calculation in the *Almagest* of the latitude ϕ of a place as a function of the length of the longest day in the year cannot be done using

$\varphi = \sin \max d_{\odot} \cot \varepsilon$, and the equivalent with the chords of the most general formula, $\sin d = \tan \phi \tan \delta$, is applied several times to the determination of the equation of the day, d.[24] Another calculation facilitated by the table of tangents is that of the angle of a plane right-angled triangle of which the sides of the right-angle are given. The problem occurs with regard to planetary models, in particular in the construction of tables of equations. Furthermore the calculation of the sun's altitude from the gnomon's shadow involves the use over and over again of Pythagoras's theorem to extract a square root, and we have seen before the increasing role of local co-ordinates in Arabic and Indian astronomy.

In the first Arabic texts, the calculation of the sun's altitude still works through the hypotenuse of the triangle determined by the gnomon and its shadow, the 'diameter of the shadow'. For a vertical gnomon g, with a shadow o on a horizontal plane, the altitude of the sun, $h_{\odot}$, is calculated by $\mathrm{Sin}\ h_{\odot} = Rg/d$, with $d=(g^2+o^2)^{1/2}$. We can quickly tabulate the shadow as a function of altitude. Most frequently the gnomon is divided into 12 fingers, following an Indian tradition. There are other divisions, for instance in $6\frac{1}{2}$ or 7 feet or in 60 parts. A table of the shadow as a function of altitude, in two places, by degrees, for a gnomon of 12 fingers—thus corresponding to the function $\theta \rightarrow 12 \cot \theta$—can be found in the *zīj* of al-Khwārizmī (the author of the famous algebra, Baghdad, beginning of the ninth century) and of al-Bāttānī (Raqqa, end of the ninth century). In the two treatises it is only applied to reciprocal calculations of the altitude and the shadow. The gnomon, a magnitude as arbitrary as the radius of the sphere, with its own units, is clearly an obstacle to any idea of generalization and the introduction of a useful function, tan or R tan. In the same period, Ḥabash al-Ḥāsib does not have a table in his *zīj* for the shadow of the sun. He calculates the altitude by the traditional 'diameter of the shadow', for a gnomon of 12 fingers. It is in this treatise, however, without doubt one of the most important that have been preserved from the astronomy of the ninth century, that the general notion of the tangent of an arc appears, with a definition, a table and several applications. The way in which Ḥabash introduces it makes us think that he did not take it from a predecessor. Apart from the difficult question of priority or of the status which it is fitting to give to the tables of the sun's shadow, the text of Ḥabash is of the first importance for us. It is a context which explains the introduction of the new function and, paradoxically, the low interest that it has created before occupying in the *zīj* a place comparable with that of the sine function.

Aḥmad b. ʿAbdallāh Ḥabash al-Ḥāsib al-Marwazī belongs to this generation of astronomers who discovered the *Almagest* after having been educated in Indian methods. A contemporary of both al-Khwārizmī and al-Battānī who had their works translated into Latin, Ḥabash remained almost unknown in the medieval West. It is mainly from the work of al-Bīrūnī, an infinitely valuable and reliable source, that historians had their attention drawn to this author, often cited in reference. Of the writings of Ḥabash, concerning the whole of astronomy, hardly any survived—and were unfortunately in some disorder: his *Verified table,* i.e. his *zīj* called *mumtaḥan*, which he wrote towards the end of his life, at least after 869. To him alone this text justifies the surname al-Ḥāsib (the mathematician) given to the astronomer of Baghdad. Although it concerns an astronomy treatise formed of rules and tables, with their proper composition, and not a commentary to the *Almagest,* the work proceeds on the basis of consideration of the less obvious mathematical ideas in Ptolemy's book. As an example, we can quote the application made by Ḥabash of the formula

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

of which the geometrical equivalent is used by Ptolemy to construct a table of chords and from which he extracts an original procedure to find the square root using the sine table. Everywhere Ḥabash tries to improve the techniques in the *Almagest*. We see him fill the blanks in the spherical calculation, develop interative procedures, using also Indian sources, and expand the use of interpolation functions which can be found in the tables of equations. Perhaps it is this last very interesting idea of Ptolemy that he used to connect the elaboration of his famous *Jadwal al-taqwīm* (table of the exact positions…), which we are now going to talk about before we come back to the tangent function.

The interpolation functions of the *Almagest* are an artifice applied to some particular functions of two variables to reduce, to a good approximation, the role of the less influential of the two extreme values and thus to avoid an irksome tabulation.[25] The four functions that compose the *Jadwal al-taqwīm* of Ḥabash proceed likewise from the treatment of the expressions of two parameters. No explanation was given for the construction of the table but this is clear from the similarity between its main applications. It is thus that in the model

$$\text{Sin}\ \delta = \frac{\text{Sin}\,[\beta + f_1(\lambda)]\,.f_2(\lambda)}{R} \qquad \text{Sin}\ \Delta\alpha = \frac{f_3(\lambda).f_4(\delta)}{R}$$

which serves to determine the equatorial co-ordinates *(α, δ)*[26] of a heavenly body with given ecliptic co-ordinates *(λ, β),* Ḥabash not only makes the reciprocal change of co-ordinates but also calculates the complement, $\bar{\nu}$, of the angle ecliptic/horizon and the equation of the period of the sun, $d_\odot$, directly as a function of the primitive data, the latitude ϕ, the sidereal time α_M and the longitude $\lambda_\odot$, through

$$\text{Sin}\ \bar{\nu} = \frac{\text{Sin}\,[\varphi - f_1(\alpha_M)]\,.f_2(\alpha_M)}{R} \qquad \text{Sin}\ d_\odot = \frac{f_3(\lambda_\odot - 90^\circ).f_4(\varphi)}{R}$$

At the origin of the analogy, perceived in one way or another, there is the possibility of placing the elements of the problem in the same configuration formed by the ecliptic, the equator and two great perpendicular circles passing through a heavenly body or the zenith of a place. From this results also the remarkable character of all the functions constructed by Ḥabash. Contrary to the usual tables of astronomy treatises which give the result of a particular calculation, or a calculation step, these auxiliary functions are applicable to different variables, resulting in a major simplification compared with that resulting from the use of simple trigonometric functions, as they introduce the obliquity of the ecliptic. Their definition is not given by Ḥabash. The fourth, f_4, is identified except for a factor with the tangent function.[27]

The *Jadwal al-taqwīm* of Ḥabash was commentated and imitated by Arab authors. Abū Naṣr, whom it inspired in his *Jadwal al-daqā'iq* (table of minutes), an ensemble of five auxiliary functions, makes a complete analysis of the table and its applications.[28] He cites a commentary from al-Khāzin and a table of the same kind from al-Nayrīzī. Created by the development of spherical calculation, the research on auxiliary functions, of which the *Jadwal al-taqwīm* is probably not the first essay, continues in several different ways. Among the tabulated functions, some are purely trigonometric, like the inverse of the sine, formerly called the cosecant. This means that despite the possible applications to Menelaus's theorem or to certain resolutions of plane triangles, the tangent, when it is introduced, can hardly be distinguished from the other auxiliary functions. In the *zīj* of Ḥabash, the notion of the 'shadow' of an arc, qualified as 'very useful', is presented in a short rubric for a change in co-ordinates. Ḥabash uses the first two first examples of calculation to define the 'shadow' (called here Tan) of the latitude ϕ of a place and that of its complement $\bar{\varphi}$ by the equivalent of

$$\text{Tan } \varphi = \frac{R \text{ Sin } \varphi}{\text{Sin } \bar{\varphi}} \qquad \text{and} \qquad \text{Tan } \bar{\varphi} = \frac{R \text{ Sin } \bar{\varphi}}{\text{Sin } \varphi}$$

This definition on a particular case is not in the same style as the accounts dedicated elsewhere to the sine and to the versed sine. The notion itself is general, as the table is applied to the resolution of several problems of which one, plane, relates to the sun's equation. The 'table of the shadow' (function *R* tan) of the *zīj* of Habash is for three places, by half-degrees from 0; 30° to 89°.

In the period of the *Keys,* the tangent of an arc is not always part of the usual ideas. It is certain that there were authors like Ibn Yūnus (d. 1009), who paid no attention to the function tabulated by Ḥabash. In his Great *Hakemite Table,* where the spherical calculation holds such an important place, the astronomer from Cairo returns to the table of the sun's shadow (function $h_\odot \mapsto g \cot h_\odot$) and to the only reciprocal calculations of the shadow and the altitude. When he presents the tangent rule, in his *Keys,* al-Bīrūnī has to explain what he understands by 'shadow' of an arc, although he does not mention any definition concerning the sine. He does it by modifying the exposition of Abū al-Wafā', in order to distinguish better the 'shadow' of an arc from the two kinds of shadow of a gnomon (Figures 15.10 and 15.11).[29] We also see, in this period, al-Khujandī and Kūshyār, the two astronomers who met in Rayy, rejecting the tangent theorem of Abū al-Wafā', objecting that the use of the table of shadows is incorrect because of the rapid increase in their difference, a variation concretized by the lengthening of the shadows of the gnomon. This term 'shadow', borrowed from gnomonics, is in effect charged with meaning, as is shown in the book of al-Bīrūnī written about twenty years after the *Keys,* his treatise on *Shadows,*[30] which collects all sorts of considerations on shadows and their measurements, applied to the trace of the line of the meridian, to the determination of the hours for prayer, to the evaluation of distances

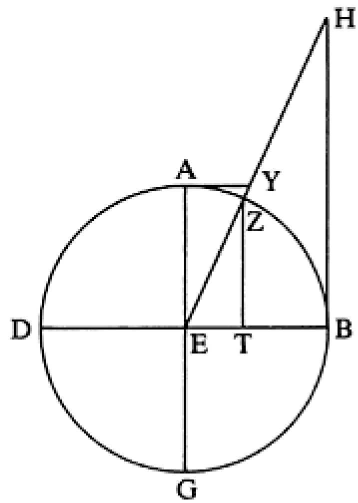


Figure 15.10

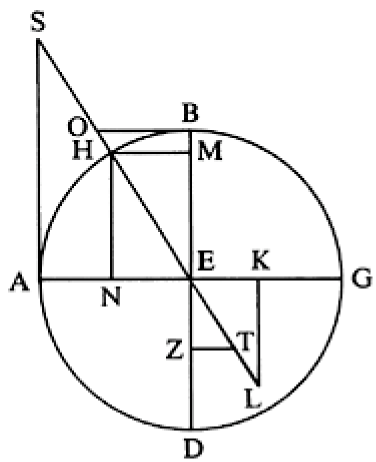


Figure 15.11

etc., before stating the simplifications that they bring to astronomical calculation. Nevertheless, Abū al-Wafā', the work of about 'shadows' considers only the tangent function.

The fifth chapter of the first book of the *Almagest* of Abū al-Wafā', treats sines and chords. The sixth is dedicated to 'shadows', 'because of the necessity of using them in most questions'. Abū al-Wafā', geometrically defines the 'shadow' of an arc, which he calls his 'first shadow' or his 'versed shadow', identifying it to the versed shadow of a horizontal gnomon mistaken for a radius of the reference circle (in our notation, Figure 15.10, $\mathrm{Tan}\,\widehat{BZ} = BH$, the shadow of BE). The same figure is used to introduce the 'second shadow', or 'extended shadow', of the arc considered ($\mathrm{Cot}\,\widehat{BZ} = AY$, the shadow of AE) and to establish all the elementary relations between tangent, cotangent, sine and sine of the complement, some of them expressed with the help of two 'shadow diameters' (respectively EH and EY), our old secant and cosecant. al-Wafā' also notes that taking the norm as unity, the 'shadow' will be equal to the ratio of the sine to the sine of the complement, and the same for the 'second shadow'. His exposition from now on classical is the counterpart to the geometric definitions of the sine and the versed sine. He starts in the *zīj* with the tangent rule and the table of the 'shadow'. The word *ẓill* (shadow) has been preserved in Arabic to mean what we call the 'tangent', a term quite ambiguous and therefore criticized by Viete when it was introduced by Thomas Fincke (1583). Before him, Maurolycus, who translated the *Spherica* of Menelaus from Arabic, uses the term *umbra versa* in his *De sphaera sermo* (1558), notably with regard to the tangent theorem. However, we do not know exactly by what routes the use of the tangent function was introduced in the West.

THE TREATISES ON TRIGONOMETRY

The end of the tenth century truly marks a turning point. In astronomy texts, trigonometry occupies an important place, with chapters on sines and chords, 'shadows' and the formulae of spherical calculation. There is also an interest in the resolution of triangles. The study of the triangle replaces somewhat the traditional developments on Menelaus's theorem, which it still encompasses to form the substance of a new type of work, represented by the *Treatise on the Quadrilateral* of Naṣīr al-Dīn al-Ṭūsī. This research is accompanied by some aquisitions, like the latest relations of the right-angled spherical triangle or the notion of polar triangle. It will not enrich the calculation of the *zīj* with new procedures. The expression of a technique which has already fulfilled its own objectives, the trigonometry of the treatises is essentially spherical and places the right-angled triangle in a place of honour.

The question of the resolution of spherical triangles begins to distinguish itself from the astronomical context during this period which precedes the introduction of the triangle formulae. A piece written by Abū Naṣr about the mistakes made by Abū Ja'far al-Khāzin in his *Zīj al-Ṣafā'iḥ* seems even to indicate that al-Khāzin had achieved the resolution of ordinary triangles in most cases, including that of three sides given.[31] With the contribution of new theorems, the question is obviously rephrased. Al-Bīrūnī develops it in his *Keys* in order to prove that all spherical calculation is possible by the only 'figure that dispenses [from the quadrilateral]'. He first divides the triangles into ten classes according to the nature of their angles, establishes some properties relative to the sides, and then joins the classes to retain the one formed by triangles which have one right-angle and two acute angles. He solves the right-angled triangles by means of some formulae combined under the term *al-shakl al-mughnī,* mentioning occasional simplifications introduced by the other 'figure', *al-shakl al-ẓillī* (Figure 15.12).[32] His solution of any triangle, by decomposition into

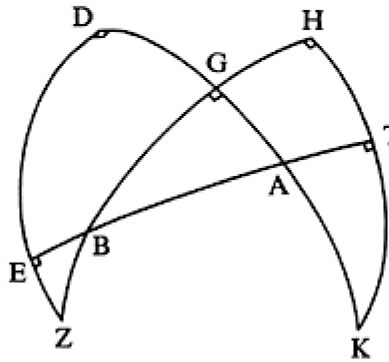


Figure 15.12

two right-angled triangles with the help of an altitude, is more basic. The two cases of three sides given or three angles given are missing. Considered by itself, the study of al-Bīrūnī has some gaps. The aim is different, which justifies the applications to astronomy. Nevertheless, the idea will be taken up again. From the *Keys,* the Arab authors retain Menelaus's theorem, the ' figures' which replace it, the classification of the triangles and their resolution. These are the elements entering in the composition of several purely mathematical works leading to the *Treatise on the Quadrilateral* and situated to the side of the astronomical calculation.

The six relations of the right-angled triangle appear in an anonymous text at the end of the eleventh century, a compilation which does not have the quality of the book by Nasīr al-Dīn but which already heralds the plane.[33] In fact, the author of this treatise establishes fourteen formulae more or less matching each other of which he does not make any use. On the other hand he reports an interesting resolution of the triangle due to Abū Naṣr. Two written pieces[34] by him in effect complete the statements in the *Risāla*. The first one, which assembles the questions treated at the request of al-Bīrūnī, contains the plane theorem of the sine, the statement of which is suggested by the spherical theorem: 'when you knew'—writes Abū Naṣr 'that in the triangles formed by arcs of great circles of a sphere, the ratio of the sine of one side to the sine of the other side was equal to the ratio of the sine of the angle opposite to the first side to the sine of the angle opposite to the second, you asked if the rule was general for all triangles, I mean that they were formed by arcs or straight lines. Our answer is yes…' The theorem—in the equivalent form $g/b=\text{Sin } G/\text{Sin } B$—is then demonstrated with the help of a sketch of an altitude (Figure 15.13).[35] The second written piece, from which is drawn the method reported in the anonymous text, is precisely that in which Abū Naṣr corrects the mistakes of al-Khāzin. It is important because we can find there the first utilization of the polar triangle.

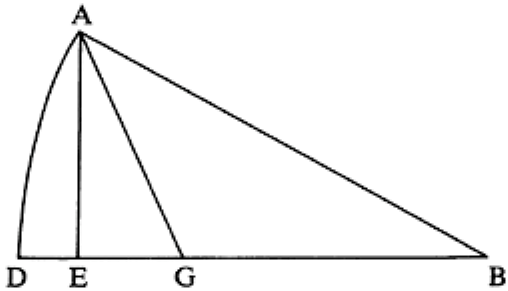


Figure 15.13

The use of the polar triangle, applied to the resolution of an ordinary triangle with given angles, was first noted in the *Treatise on the Quadrilateral* (1260). It was the first known use of the duality principle developed in Europe in the time of Viete (1593). Perhaps we could have observed that Naṣīr al-Dīn lost some opportunities to apply the idea introduced, notably in his study about the respective nature of the sides and angles of a triangle.[36] With knowledge of the anonymous text, the idea was thus attributed to the author of this treatise at the end of the eleventh century. We know now that the method exposed by Naṣīr al-Dīn is due to Abū Naṣr and that it goes back to a construction by al-Khāzin.[37] We find ourselves taken to the tenth century, i.e. to the period when this kind of question was worked out and made clearer. Al-Khāzin, who was interested in various subjects, is known as the original author, sometimes negligent. His calculation, effectively wrong, has the merit of putting the problem well, by the construction of arcs measuring the angles of the initial triangle. Abū Naṣr rectifies the figure and completes it, showing a triangle (HKS, Figure 15.14) in which he proves that the angles and the sides are the respective supplements of the sides and the angles of the initial triangle (ABG). The problem is then brought back to the determination of the angles of a triangle of given sides, which he has solved at the beginning with the help of his own triangle formulae. The written piece in which Abū Naṣr introduces this remarkable figure lends itself poorly to other developments, which does not suggest, however, the only theorem of the sine, invariant by duality. It does not seem that the Arab authors had made other uses of the polar triangle. We only know one other construction, more complicated, applied to the same problem and accomplished from the poles of sides of the initial triangle. The method can be found in a trigonometry treatise probably written in Spain at the beginning of the eleventh century. Its author Ibn Muʿādh explains the difficulty

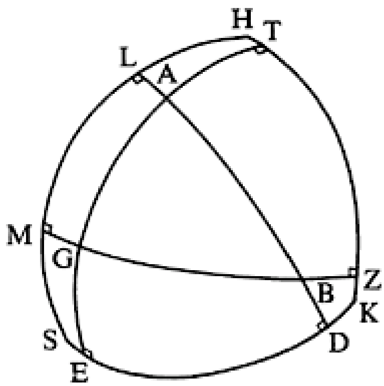


Figure 15.14

he encountered when solving this problem. We shall come back to the interesting treatise of Ibn Muʻādh after we have studied the composition of the book of Naṣīr al-Dīn.

Naṣīr al-Dīn al-Ṭūsī (1201–74), the founder of the famous observatory of Marāgha, lived in a period which saw the collapse of the Abbasid caliphat at the same time that it brought to Islam a certain openness to the Far Eastern world. He is the author of more than sixty works some of which concern philosophy or theology. His scientific work, which includes numerous revisions of his predecessors' works, is presented as a sort of actualization of the mathematical and astronomical corpus. The *Treatise on the Quadrilateral,*[38] in five books, is included in this vast synthesis which encompasses the *Elements,* the *Spherica,* the *Almagest* and many other Greek works as well as Arab works. Books I, II and IV concern compound ratios and Menelaus's theorem, both plane and spherical, giving place to multiple inventories in an effort at exhaustivity. Book III, about the lemmas necessary for spherical calculation, briefly evokes the resolution of plane triangles using only the sine theorem.[39] It is book V in particular which constitutes the proper trigonometric part. The first four chapters take up again, and complete, the classification of al-Bīrūnī. We can find there, after comparing the elements of triangles formed by the intersections of three great circles, a double repartition of the spherical triangles into ten classes according to the nature of their sides or their angles and, for each, the study of each class of triangles and of the intersections to which they give rise.

Chapters 5 and 6, dedicated to the relations of the right-angled triangle, offer a comparable symmetry, from the two fundamental theorems, *al-shakl al-mughnī* and *al-shakl al-ẓillī.*same in the two chapters: a statement of the main theorem, demonstrations classified by type, the eventual extension to any triangle and corollaries. The six fundamental relations, expressed in a general form, are applied in Chapter 7 to the resolution of right-angled triangles, carried out again either by the formulae in Chapter 5 or by those in Chapter 6. Symbolically translated (for a triangle ABG, right-angle at G), and with the numbering of Braunmühl, the formulae established by Naṣīr al-Dīn are as follows.

Chapter 5: the relation

$$\frac{\text{Sin } g}{R} = \frac{\text{Sin } a}{\text{Sin } A} \qquad \text{(I)}$$

with its extension to any triangle and, as a corollary, the relations

$$\frac{\text{Cos } a}{\text{Cos } g} = \frac{R}{\text{Cos } b} \qquad \text{(III)}$$

$$\frac{\text{Cos } A}{\text{Cos } a} = \frac{\text{Sin } B}{R} \qquad \text{(V)}$$

the last in two variants, the formulae (3) and (4) of Abū Naṣr.

Chapter 6: the relation

$$\frac{\text{Sin } a}{R} = \frac{\text{Tan } b}{\text{Tan } B} \qquad \text{(II)}$$

which cannot be generalized like relation (I) and which has as corollaries

$$\frac{\text{Cos } A}{R} = \frac{\text{Cot } g}{\text{Cot } b} \qquad \text{(IV)}$$

$$\frac{\text{Cos } g}{R} = \frac{\text{Cot } A}{\text{Tan } B} \qquad \text{(VI)}$$

as well as two statements analogous to (3) and (4).

The work is completed, Chapter 7, with the resolution of any triangle, reduced to that of right-angled triangles and including the use of the polar triangle described earlier. The *Treatise on the Quadrilateral* is a remarkably constructed book which obviously takes up a traditional style. We know two treatises that are less elaborate prior to the book of Naṣīr al-Dīn and with the same contents—the anonymous text already mentioned and another text, next to the *Keys* of al-Bīrūnī. The *Book on the Unknown of the Arcs of the Sphere,*[40] by Ibn Muʿādh, is not situated exactly in this filiation.

Against the well-organized *Treatise on the Quadrilateral,* the patchy work of Ibn Muʿādh makes a singular contrast. The ideas are linked together haphazardly, and the author is not afraid occasionally to go back to an essential point or one forgotten. The recent discovery of this small original treatise brings us more questions than answers about the difficult question of transmission to the West. From a family of lawyers in Andalusia, the qāḍī Abū ʿAbd-Allāh Muḥammad ibn Muʿādh al-Jayyānī (989 to after 1079) had stayed in Cairo during his youth (1012–16), where he was probably the student of Ibn al-Haytham. He left some work of quality which caused him to be considered, in Spain, as one of the best mathematicians of his generation. Several of his works were translated into Latin, but we cannot find any trace of the influence of this *Book on the Unknown,* itself very different from Eastern treatises. From the beginning of the eleventh century, many texts circulated with the new calculation techniques for astronomers. It is probably with a fragmented knowledge about the progress accomplished by the scholars of the Middle East that Ibn Muʿādh wrote this book, basing his thoughts on the *Spherica* of Menelaus, the only work quoted in reference. The six relations of the right-angled triangle are established from Menelaus's theorem, the same figure being used to extend relation (I) to any triangle. The solution of triangles is split into as many cases as the determinations of unknown elements, with a rigorous discussion about the nature of the arc obtained by its sine or its tangent. The polar triangle is used in the case of three angles given. For the tangent, which he tabulates, Ibu Muʿādh does not use the term 'shadow'. It seems that he reintroduces the notion, under a slightly different form, of the 'ratio of the sine to the sine of the complement' (function tan, not *R* tan). He does this in the course of the calculation that we are now going to discuss.

A common point in all these treatises is the almost complete absence of plane trigonometry. One calculation is necessary, the determination of two arcs knowing their sum or their difference and the ratio of their sines. Naṣīr al-Dīn explains two methods, one taken from *Almagest* where the problem is dealt with using chords, and the other attributed to Abū Naṣr, both of them employing Pythagoras's theorem. Ibn Muʿādh poses the problem—in the case of the difference—in a form equivalent to

$$\frac{\text{Sin } x}{\text{Sin } y} = \frac{a}{b} \qquad x - y = \alpha, \text{ with } a > b;\ 0^\circ < \alpha < 180^\circ$$

the unknown arcs x and y lying between 0 and 180º. After having established the uniqueness of the solution geometrically, he gives us a construction explaining the reason for choosing $a \neq b$. He then deals with the particular case $\alpha=90°$ in order to introduce the tangent and finally deduces from the figure the equality

$$\tan\left(\frac{x+y}{2}\right) = \frac{a+b}{a-b}\tan\left(\frac{\alpha}{2}\right)$$

which gives x and y by their sum and their difference. The method of Ibn Muʿādh, interesting for the final calculation, is representative of the techniques employed: geometrical elaboration of the algorithm, and then the description of the calculation 'independently of the figure'. The formula is not obtained by transforming the relation

$$\frac{\text{Sin } x + \text{Sin } y}{\text{Sin } x - \text{Sin } y}$$

but with the help of a similarity, the required tangent representing the ratio of the sides of the right-angle of a right-angled triangle (Figure 15.15).[41] This kind of trigonometric application, appealing to the geometric meaning of the sine or the tangent, can be found in very diverse texts, notably for the evaluation of distances. The *Key of Arithmetic* of al-Kāshī, a compendium about calculation techniques compiled in the fifteenth century, contains a small table of sines for solution of plane triangles and for formulations in relation to the measurement of surfaces, such as

$$r = \frac{\text{bg Sin A}}{60(a+b+g)}$$

giving the radius of a circle inscribed in a triangle (ABG).

The fundamental formulae of plane trigonometry are found in astronomy texts, where they are applied to the construction of the sine table. Chapter I, 5 in the Almagest of **Abū al-Wafā'** is a good example. We extract the first six sections which contain the definitions and the formulae. **Abū al-Wafā'** first describes the defining segments: diameter, chord, 'extended' sine or sine, versed sine or 'arrow', sine of the complement (called here Cos), chord of the 'complement' (i.e. Crd(180°−α)) and the larger sine *(R),* for *R*=60. After a short section about the rational sine and chords, he successively studies

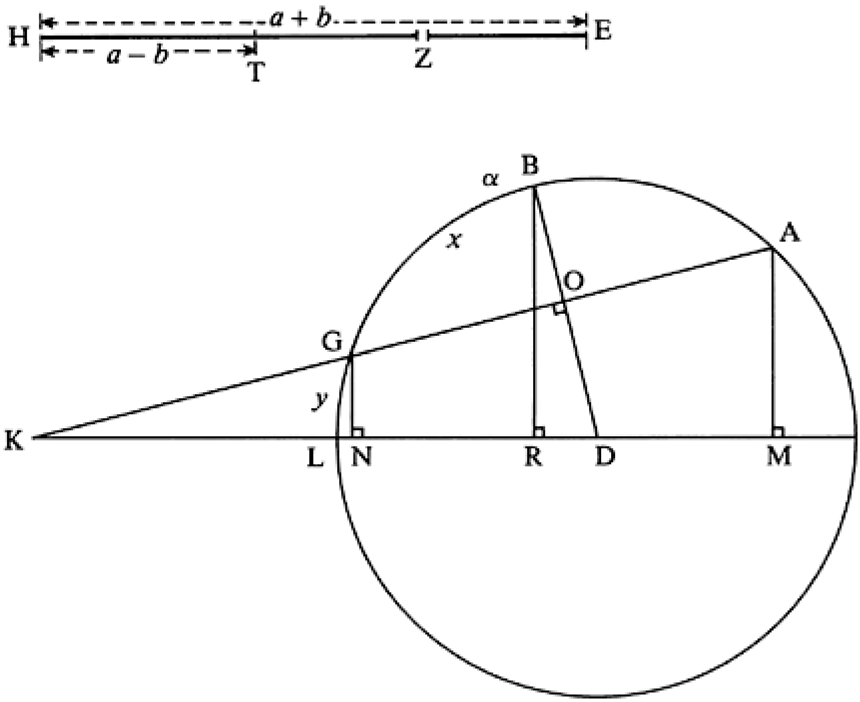


Figure 15.15

the determination of sines and chords of complements, the reciprocal calculation of the sine and the chord, the determination of sines and chords of the halves and the doubles, and then the sums and the differences. Some formulae are therefore applied to reciprocal calculations. They are all geometrically established and accompanied by examples. In regrouping the equivalent formulae obtained by the same construction, the expressions are as follows.

(A) Section 3

(a) Cos $\alpha=(R^2-\mathrm{Sin}^2\alpha)^{1/2}$ and $\mathrm{Crd}(180°-\alpha)=(4R^2-\mathrm{Crd}^2\alpha)^{1/2}$

(b) Vers $\alpha = R \mp \mathrm{Cos}\ \alpha \qquad (\alpha \lesseqgtr 90°)$

(c) $[\mathrm{Vers}\ \alpha\ (2R-\mathrm{Vers}\ \alpha)]^{1/2}=\mathrm{Sin}\ \alpha$

(B) Section 4 and beginning of section 5

$$\frac{\mathrm{Vers}\ \alpha}{\mathrm{Crd}\ \alpha}=\frac{\mathrm{Crd}\ \alpha}{2R}, \qquad \frac{\frac{1}{2}\ \mathrm{Vers}\ \alpha}{\mathrm{Sin}\ (\alpha/2)}=\frac{\mathrm{Sin}\ (\alpha/2)}{R}$$

and

$$\frac{2R-\mathrm{Crd}(180°-\alpha)}{\mathrm{Crd}\ (\alpha/2)}=\frac{\mathrm{Crd}\ (\alpha/2)}{R}$$

(C) End of section 5

$$\frac{\mathrm{Crd}\ \alpha}{\mathrm{Crd}\ (\alpha/2)}=\frac{\mathrm{Crd}(180°-\alpha/2)}{R}$$

where it is deduced that

$$\frac{1}{2}\ \mathrm{Sin}\ 2\alpha=\frac{\mathrm{Sin}\ \alpha\ \mathrm{Cos}\ \alpha}{R}$$

(D) Section 6

(a) (See Figure 15.16)[42]

$$\begin{cases} \mathrm{Sin}(\alpha \pm \beta)=\left(\mathrm{Sin}^2\ \alpha-\dfrac{\mathrm{Sin}^2\ \alpha\ \mathrm{Sin}^2\ \beta}{R^2}\right)^{1/2} \pm \left(\mathrm{Sin}^2\ \beta-\dfrac{\mathrm{Sin}^2\ \alpha\ \mathrm{Sin}^2\ \beta}{R^2}\right)^{1/2} \\ \mathrm{Sin}(\alpha \pm \beta)=\dfrac{\mathrm{Sin}\ \alpha\ \mathrm{Cos}\ \beta}{R} \pm \dfrac{\mathrm{Sin}\ \beta\ \mathrm{Cos}\ \alpha}{R} \end{cases}$$

(b) Two analogous expressions with chords.

Section 7 relates to ‘mother’ chords,[43] and all the rest of the chapter concerns the sine table.

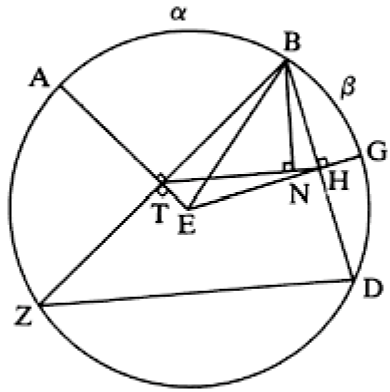


Figure 15.16

The very methodical exposition of Abū al-Wafā' is, indeed, particularly developed, overburdened by the use of the versed sine and the chord. These rules, stemming from the *Almagest,* are considered as a set which, in the *zīj,* is disassociated from the subject of 'shadows'. They constitute, for instance, one of the rubrics of the very geometrical treatise of al-Bīrūnī on *Chords,*[44] dedicated to some theorems related to a broken line inscribed in a circle. In the *Qānūn,* al-Bīrūnī keeps the simplification R=1 already introduced by Abū al-Wafā'. More than the retention of R=60, commonly adopted and very convenient for the tables, the absence of negative numbers limits and slightly complicates the use of these formulae. It remains that, with a tangent and with the formulation of fundamental relations and the support of algebraic techniques, Arab mathematicians had at their disposal the necessary bases for the development of trigonometric calculation. Hindered undoubtedly by the recourse judged necessary to geometrical proof, their researches were not oriented in this direction. Instead it is in the work of perfecting the tables that algebra and trigonometry meet up in some way.

THE SINE TABLE

All the precision of astronomical calculation is based on the exactitude of the sine table. Its construction is linked with the famous problem of the trisection of an angle. The research, which was ongoing from the tenth century, is inscribed in the more general framework of approximation calculations applied to certain categories of irrationals. Rigorous or simply intuitive, these approaches were analysed by the historians. They are interesting because of the techniques used: interpolation techniques or algorithmic procedures. The tables of the *zīj* are generally more precise than the table of chords in the *Almagest,* without reaching the precision of the tables which were constructed in Europe not long before the introduction of logarithms.

The table of chords in the *Almagest* is sexagesimal to three places, with entries at half-degrees. It is exact, and it is easy to verify that, by linear interpolation, it gives the sine arc to within some seconds except in the neighbourhood of 90° the error exceeding a minute of arc above 89;45°. In the ninth century, the Indian tables were already known, and they did not involve the same degree of approximation. The precision was apparently considered sufficient. Habash, whom as we have seen adopts most of the calculations of the *Almagest,* transposes the table of chords just as it is; for the sine, it has entries every fifteen minutes and a fourth column formed from 0 and 30. Al-Bāttānī simplifies this, keeping half of the table (entries by half-degrees) and suppressing the numbers in the fourth place. The texts do not indicate whether the sine table was calculated before the end of the tenth century. The two first known original constructions are due to Ibn Yūnus and Abū al-Wafā', that of Ibn Yūnus stemming more directly from the procedure in the *Almagest*. Ptolemy, remember, determines a chord to 1° by bracketing, using a theorem established by comparison of two areas which expresses the decrease, between 0° and 90°, of (Sin x)/x in the form

$$a > b \Rightarrow \frac{\text{Crd } a}{\text{Crd } b} < \frac{a}{b}$$

He deduces the double inequality

$$\frac{2}{3}\text{ Crd } 1;30^\circ < \text{Crd } 1^\circ < \frac{4}{3}\text{ Crd } 0;45^\circ$$

and then, approximately,

$$\frac{2}{3}\text{ Crd } 1;30^\circ = \text{Crd } 1^\circ = \frac{4}{3}\text{ Crd } 0;45^\circ = 1;2,50$$

The calculation of chords of 1;30° and 0;45°, made using the sides of an inscribed regular pentagon and an inscribed regular hexagon, through Crd(72°−60°) followed by four bisections, could have been conducted with superior precision by considering the difference between the two values. It is clear that Ptolemy chose in contrast the amplitude of the interval (3/4° for the chord, i.e. 3/8° for the sine) in order to obtain the equality to the precision required.[45]

The sine table constructed by Ibn Yūnus in his *Great Hakemite Table*[46] is sexagesimal to four places with entries every sixth of a degree. The method used is interesting in particular because of the interpolation formula which is used to complete the table from values calculated separately for half-degrees. Independently of this aspect, to which we are going to return, some modifications are introduced to Ptolemy's calculation. Ibn Yūnus starts by halving the interval he chooses for Sin 1°. He does the calculations at five places, through four bisections from Sin 15° (=Sin(45°–30°)) and Sin 18° (a half-side of the decagon), obtaining:

$$\frac{8}{9}\text{ Sin } \frac{9^\circ}{8} < \text{Sin } 1^\circ < \frac{16}{15}\text{ Sin } \frac{15^\circ}{16}$$

i.e.

1; 2, 49, 40, 4<Sin 1°<1; 2, 49, 45, 10.

He then deduces a first value of Sin 1° by

Sin 1°=1; 2, 49, 40, 4+2/3 of the difference
=1; 2, 49, 43, 28

which corresponds to the linear interpolation:

$$\frac{\text{Sin } \frac{16^\circ}{16}}{\frac{16}{16}} = \frac{\text{Sin } \frac{18^\circ}{16}}{\frac{18}{18}} + \frac{2}{3}\left(\frac{\text{Sin } \frac{15^\circ}{16}}{\frac{15}{16}} - \frac{\text{Sin } \frac{18^\circ}{16}}{\frac{18}{16}}\right)$$

Finally, he applies a small correction to the value obtained, based on the approximation that the error on Sin 1° affects, in equal amounts but with opposite sign, Sin 2.1° and Sin(3°−1°). He finally obtains

Sin 1°=1; 2, 49, 43, 28–1/2(Sin 2.1°–Sin(3°–1°))
=1; 2, 49, 43, 28–1/2(2; 5, 38, 18, 0–2; 5, 38, 17, 12)
=1; 2, 49, 43, 4

(Sin 1°=1; 2, 49, 43, 11, 15 to six places).[47]

The method of Ibn Yūnus allows the precision required to be easily attained, but careless calculations make the table inexact, the error sometimes exceeding unity in the number in the fourth place.

The way of determining Sin 1/2° used by Abū al-Wafā'[48] is different from the procedure in the *Almagest* and more advantageous. It also uses a slow variation in the vicinity of 1/2° of a decrease in the first differences of the sine. The thirtieths of differences are tabulated for the chords, in the *Almagest,* in order to facilitate the reading of the table by linear interpolation. The decrease in first differences put clearly in this way is geometrically verified by Theon in his *Commentary on the Almagest*. Abū al-Wafā' gives a different demonstration for the sine (Figure 15.17, with $\widehat{BG} = \widehat{GD}$):

$$\text{Sin } \widehat{AD} - \text{Sin } \widehat{AG} < \text{Sin } \widehat{AG} - \text{Sin } \widehat{AB}$$

because DY<DM<DK=GT. He deduces a bracketing of Sin 1/2° by choosing three values close to 1/2°, of known sine, represented by the points B, G, Z of a circle (Figure 15.18):

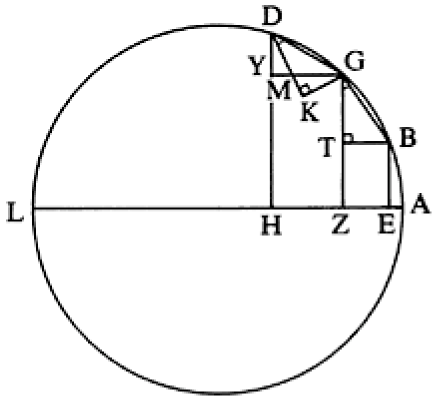


Figure 15.17

$$\widehat{AB} = \frac{3^\circ}{8} = \frac{12^\circ}{32} \qquad \widehat{AG} = \frac{9^\circ}{16} = \frac{18^\circ}{32} \qquad \widehat{AZ} = \frac{15^\circ}{32}$$

The arc BG, divided into six equal arcs, Z and the point H such that $\widehat{AH}$ is equal to 1/2°=16/32°, are part of the subdivision and the repeated application of the theorem leads to a double inequality:

$$\frac{1}{3}(\text{Sin } \widehat{AG} - \text{Sin } \widehat{AZ}) < \text{Sin } \widehat{AH} - \text{Sin } \widehat{AZ} < \frac{1}{3}(\text{Sin } \widehat{AZ} - \text{Sin } \widehat{AB})$$

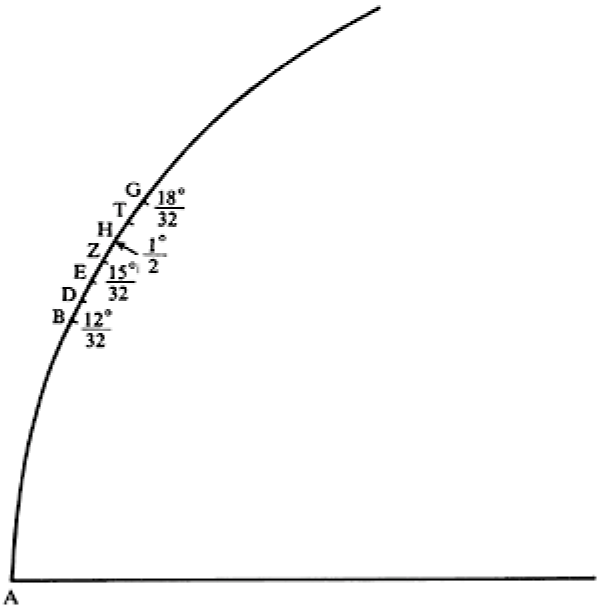


Figure 15.18

i.e.

$$\frac{1}{3}\left(\text{Sin}\,\frac{18^\circ}{32}-\text{Sin}\,\frac{15^\circ}{32}\right)<\text{Sin}\,\frac{1^\circ}{2}-\text{Sin}\,\frac{15^\circ}{32}<\frac{1}{3}\left(\text{Sin}\,\frac{15^\circ}{32}-\text{Sin}\,\frac{12^\circ}{32}\right)$$

Abū al-Wafā' thus obtains: 0; 31, 24, 55, 52, 2<Sin 1/2°<0; 31, 24, 55, 57, 47, and then, by half sum, Sin 1/2°=0; 31, 24, 55, 54, 55. The calculation is not perfectly exact,[49] but the method gives a bracketing that is almost six times finer than the procedure in the *Almagest* applied to the same values.[50] We find it again in texts until the fifteenth century. It is applied for example to the calculation of Sin 1° by Muḥyī al-Dīn al-Maghribī (thirteenth century), one of the astronomers of Marāgha at the time of Naṣīr al-Dīn, author of several studies on trigonometry. The sine table in the *Almagest* of Abū al-Wafā' was forecast to four places with entries every fifteen minutes. On the same model, the table of the *Qānūn* is practically exact. This very famous work of al-Bīrūnī gives a good idea of the precision reached at this time in trigonometric calculations. With regard to the construction of the sine table, study of the *Qānūn* opens some other perspectives. With its curious interpolation formula, we remain in the same type of approach.

The research for a better approximation than that given by linear interpolation was, it seems, a constant preoccupation with Arab astronomers, who were used to handling innumerable tables in their calculations. We know now many other formulae that were used between the tenth and fifteenth centuries.[51] The question is to know how they were worked out without any notion of graphic representation. In this respect, the rule of the *Qānūn* can be retained as an example of theoretical construction. It is described for the sine and the tangent, and then generalized to any table.[52] With standard notation

$$\Delta y_{-1} = y_0 - y_{-1},\ \Delta y_0 = y_1 - y_0, \ldots, \Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1} \quad \text{with} \quad x_0 - x_{-1} = x_1 - x_0 = \ldots =$$

the formula by which al-Bīrūnī replaces the linear interpolation

$$y = y_0 + \frac{x - x_0}{d}\,\Delta y_0$$

on the interval $[x_0, x_1]$, which is[53]

$$y = y_0 + \frac{x - x_0}{d}\left(\Delta y_{-1} + \frac{x - x_0}{d}\,\Delta^2 y_{-1}\right)$$

The obvious iteration of the procedure is emphasized by an attempt, with the help of a figure, to justify the applied interpolation to Δy_{-1}. This rule from the *Qānūn* has intrigued historians because a correct expression for quadratic interpolation, equivalent to Newton's formula to order 2, can be found in the *Khaṇḍakhādyaka*, a work that al-Bīrūnī knew well and often quotes in his writings.[54]

The nice formula in the *Khaṇḍakhādyaka* allows almost satisfactory sine values to be obtained from a more than rudimentary table, reduced to six integers.[55] With the previous notation it is as follows:

$$y = y_0 + \frac{x - x_0}{d}\left(\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{x - x_0}{d}\cdot\frac{\Delta y_0 - \Delta y_{-1}}{2}\right)$$

Geometrically, this will replace the curve in the interval $[x_0, x_1]$ by a parabola passing through the three points of co-ordinates (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1). A more elaborate form of the same quadratic interpolation, corresponding to unequal intervals $[x_{-1}, x_0]$, $[x_0, x_1]$, was applied from the tenth century to calculation of the longitude of planets.[56] There were also other formulae. We shall limit ourselves to quoting the rule of Ibn Yūnus for the sine. Over the interval $[x_0, x_2]$ it leads to a parabola passing through the points (x_0, y_0), (x_1, y_1), (x_2, y_2), with $x_0=n$ (integer), $x_1=n+1/2$, $x_2=n+1$, the table being, as we have seen, primitively constructed for all half-degrees. There again the statement of the rule demonstrates the reasoning followed, Ibn Yūnus correcting the linear interpolation made on $[n, n+1]$ of a term that is zero at the limits and which gives the exact value at the centre of the interval. Symbolically the rule is as follows:

$$\mathrm{Sin}\ x = \mathrm{Sin}\ n + (x-n)\ [\mathrm{Sin}(n+1) - \mathrm{Sin}\ n] + 4(x-n)(n+1-x) \times \left[\mathrm{Sin}\left(n+\frac{1}{2}\right) - \frac{\mathrm{Sin}\ n + \mathrm{Sin}(n+1)}{2}\right]$$

It is equivalent, in the previous notation and with $(x-x_0)/d=\xi$, for d=1/2 to the usual formula

$$y = y_0 + \xi\ \Delta y_0 + \frac{\xi(\xi-1)}{2}\ \Delta^2 y_0$$

on the interval $[x_0, x_0+2d]$.[57]

Dedicated to Mas'ūd b. Maḥmūd b. Sebüktijin (1030–40), the second Ghaznevid sultan, the *Qānūn al-Mas'ūdī* is one of the masterpieces of al-Bīrūnī. He wrote this after his stay in India, aged about 60. A very rich tome, with its eleven treatises the book overtakes the usual type of astronomy works. The third treatise, dedicated to plane and spherical trigonometry, is in ten chapters, of which one is on the determination of the side of a regular enneagon.[58] Two different geometric procedures lead, after reduction *bi-l-jabr wa-l-muqābala,* to the equations[59]

$$1 + 3x = x^3 \qquad \text{verified by} \qquad \frac{\mathrm{crd}\ 80^\circ}{\mathrm{crd}\ 40^\circ} \qquad (x = 2\cos 20^\circ)$$

and

$$x^3+1=3x \qquad \text{verified by crd } 20^\circ \qquad (x=2\sin 10^\circ)$$

i.e. to two forms of the equation of the trisection. It is in this general context that, in the following chapter, the determination of the chord of 1° is approached. Twelve problems of construction are set which, if they were solved, would make it possible to realize geometrically the trisection of any angle. The chapter finishes with four calculations of the chord of 1° of which two are based on the side of an enneagon. The idea of an equation of the third degree, proposed before in the *Qānūn,* is taken up again. Its solution through an algorithm is achieved through iterative procedures.

If we keep to the model in the *Almagest,* the determination of the real or apparent syzygy represents the kind of calculation to be treated by iteration. It is done in this way by Ptolemy. The Arabic astronomy texts furnish some other examples. To stay in the domain of trigonometry, we can cite the third method of the *Qānūn* relative to the side of an enneagon. It consists in approaching the chord of 40° by the eleventh term of the series

$$\text{crd}\left(40^\circ+\frac{2^\circ}{4^0}\right),\ \text{crd}\left(40^\circ+\frac{2^\circ}{4^1}\right),\ \text{crd}\left(40^\circ+\frac{2^\circ}{4^2}\right),\ldots,$$

which is constructed by recurrence, with the help of addition formulae, by means of

$$\text{crd}\ u_0=\text{crd}(72^\circ-30^\circ)$$
$$\text{crd}\ u_1=\text{crd}(30^\circ+u_0/4)$$
$$\ldots$$
$$\text{crd}\ u_n=\text{crd}(30^\circ+u_{n-1}/4)$$

Another interesting example can be found in the *zīj* of Ḥabash, where the problem is posed about parallax. Mathematically, it appears to be a search, over the interval [0°, 180°] for a function of interpolation that is zero at the limits and has a maximum k at a point shifted in relation to the centre of the interval, i.e. $90°-m$. The function ϕ retained by Ḥabash is such that

$$\phi(t)=k\sin\theta \tag{1}$$

with θ defined implicitly as a function of the variable t by

$$t=\theta-m\sin\theta \tag{2}$$

It is clear that the function ϕ fulfils the conditions imposed. Equation (2) put in the form $\theta=f(\theta)$

$$\theta=t+m\sin\theta$$

is solved, for any t, with the help of the series (θ_n) defined by

$$\theta_0=t \quad \text{and} \quad \theta_n=f(\theta_{n-1})=t+m\sin\theta_{n-1}$$

which converge towards the solution required.[60] This calculation by Ḥabash has been described several times both for its ingeniousness and because he introduces the equation called Kepler’s equation (equation (2)).

The elegant algorithm attributed to al-Kāshī for calculation of Sin 1° has also been well studied.[61] It applies to an equation for the trisection analogous to those in the *Qānūn*. Like the calculation by Ḥabash, it calls for a series of the form $u_n=f(u_{n-1})$. However, it uses algebraic techniques, for example the disposition in tabular form for the development of expressions of the type

$$(60x_{n-1}+q_n)^3-(60x_{n-1})^3$$

from

$$(60x_{n-1}+q_n)^3-(60x_{n-1})^3=[(q_n+3.60x_{n-1})q_n+3.60^2x_{n-1}^2]q_n$$

As we have said, Jamshīd Ghiyāth al-Dīn al-Kāshī is one of the last great personalities in Islamic science. He was a director of the important observatory at Samarkand in the era of Sultan Ulugh Beg. He also distinguished himself as a mathematician. This algorithm is only known through a commentary of the astronomical tables of Ulugh Beg.[62] It is therefore difficult to know to what extent al-Kāshī is taking from a predecessor. The commentator reports a geometric demonstration establishing that Sin 1° is the solution of the equation

$$x=\frac{x^3+15.60\ \mathrm{Sin}\ 3^\circ}{45.60} \tag{1}$$

The unknown *x,* for which we know that 1; 2<*x*<1; 3, is searched for in the form $x=q_0+60^{-1}q_1+\ldots+60^{-n}q_n$, which in base 60 is $x=q_0; q_1,\ldots, q_n$, on the condition that q_k is strictly less than 60.[63] The method consists then in determining successively the numbers $q_0, q_1,\ldots, q_n$ with the help of a convergent series, each time moving the calculation to the next number and consequently retaining a number supplementary to the numeral 15.60 Sin 3°.

More precisely, putting $x_0=q_0$, $x_1=q_0; q_1$, $x_k=q_0; q_1,\ldots, q_k$, as the terms of the series and designating by *[X]* the integer part of *X* in base 60, the calculation of al-Kāshī evolves as follows, from equation (1) which we write as

$$x=\frac{x^3+N}{D}$$

(with N=15.60 Sin 3° and D=45.60). At the first step, q_0 is obtained as an integer part of *N/D,* where $x_0=q_0$; the remainder r_0 of the division of *N* by *D,* i.e. $r_0=N-Dq_0$, is used to put equation (1) in the form

$$x-x_0=\frac{x^3+r_0}{D}$$

Then we look for a q_1 such that $x_1 = q_0; q_1 = q_0 + 60^{-1} q_1$, i.e.

$$q_1 = \frac{x_0^3 + r_0}{60^{-1} D}$$

Hence the new remainder r_1 is $(x_0^3 + r_0) - 60^{-1} D q_1$ and the equation becomes

$$x - x_1 = \frac{x^3 - x_0^3 + r_1}{D}$$

In a general way, at the (k+1)th stage, the equation to be solved is

$$x - x_{k-1} = \frac{x^3 - x_{k-2}^3 + r_{k-1}}{D}$$

from which is taken

$$q_k = \frac{x_{k-1}^3 - x_{k-2}^3 + r_{k-1}}{60^{-k} D}$$

Then

$$x_k = x_{k-1} + 60^{-k} q_k \qquad \text{and} \qquad r_k = (x_{k-1}^3 - x_{k-2}^3 + r_{k-1}) - 60^{-k} D q_k$$

The commentator confines himself to showing the calculation of the first five places from an exact value of Sin 3° to eight places. Al-Kāshī would have determined the first ten places of Sin 1° We can gain an idea of the quality of his calculations by another written piece, the famous *al-Risāla al- muḥīṭiyya* dedicated to the determination of π, where by a different procedure from that of Archimedes and showing π as the limit of

$$3.2^n \sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \ldots \sqrt{2 + 1}}}}_{n \text{ radicals}}}$$

al-Kāshī obtains very accurately the ten sexagesimal places that he was looking for with the help of an error calculation.[64] The sine table in his *zīj-i Khāqānī* is exact to four sexagesimal places, with entries every minute of arc.[65] Neglected by the ninth- and tenth-century astronomers, the precision of numerical calculation characterizes this last period represented by the school of Samarkand. It benefits from the developments in algebra, notably from the work of mathematicians like al-Samaw'al al-Maghribī or Sharaf al-Dīn al-Ṭūsī.

It would be an exaggeration to say that trigonometry did not exist before the ninth century. The sine is Indian and the fundamentals go back to the Greek era with the table of chords and Menelaus's spherical theorem. The Arab scholars trusted this codified knowledge illustrated by the *Treatise on the Quadrilateral*. In their hands, the geometric calculations of the *Almagest* in the table of chords became an instrument of remarkable flexibility, whilst many other techniques in astronomical calculation were developed such as the use of auxiliary functions, interpolation and iterative procedures. The tangent function, the first relations of the triangle and the notion of the polar triangle are among the acquisitions of the science of this period. In this particular domain, full of astronomical activity, one can rediscover the steps of the Arab mathematicians, proceeding from their reading, continually renewed, enriched and reoriented by the ancient texts. Thus they managed to break away into a new discipline, which would demand some further developments before it became an indispensable element in mathematical calculation.

NOTES

1 A method which consisted in replacing the multiplication by addition with the help of formulae such as $\cos a \cos b = 1/2[\cos(a+b) + \cos(a-b))$. See for instance *Dictionary of Scientific Biography,* art. Werner J. or Wittich (1976, XIV, 272–7 and 470–1).

2 Indicating by γ, H and E respectively the ascendant node and the points of the ecliptic and of the equator situated on the eastern horizon at a given moment, the oblique ascension of 'degree' H, of longitude $\widehat{\gamma H}$, is the length of the arc of the equator $\widehat{\gamma E}$ which 'rises' above the horizon at the same time as $\widehat{\gamma H}$. At a point on the terrestrial equator, it is our right ascension. For all these arcs see the definitions in Neugebauer (1975:34–5).

3 The spherical theorem of Menelaus has for us a form analogous to that of the plane theorem, applied to all quadrilaterals formed by arcs of great circles. With the labelling of Figure 15.3, considering the triangle WEA and the secant BDG, it follows that

$$\frac{\text{Sin } \widehat{BA}}{\text{Sin } \widehat{BE}} \cdot \frac{\sin \widehat{GE}}{\sin \widehat{GW}} \cdot \frac{\sin \widehat{DW}}{\sin \widehat{DA}} = 1$$

The ancient authors did not have this representation of a triangle and of a transversal (cf. note 11 below). On the theorem of Menelaus and the formulae that can be deduced from it, the reader can consult Braunmühl (1900: especially vol. 1, pp. 24–5), or Neugebauer (1975:26–9).

4 Cf. Figure 15.1(a), quadrilateral ZBAD and secant AGD or right-angled triangle ABG.

5 Figure 15.1(b) illustrates the Indian method for the preceding rule, $\sin \delta_\odot = \sin \lambda_\odot \sin \varepsilon$, and also for another formula relative to the right ascension and distinct from

that in the *Almagest,* sin $\alpha_\odot = \sin \lambda_\odot \cdot$ cos ε/cos $\delta_\odot$.

6 More complex problems, which cannot be described here; cf. al-Bīrūnī, *Maqālīd,* pp. 37–8.

7 About parallax for instance, see Neugebauer (1975:116) and Kennedy *et al.* (1983:173).

8 The reader will find the description of one of these analemmas in the chapter dedicated to the *qibla* (method of Ibn al-Haytham). Compare also Kennedy *et al.* (1983:621–9; translation of a text by al-Bīrūnī about an analemma of Ḥabash

9 In other words formulae that can be obtained either with the help of a figure in space, as described in note 5, or with the help of the analemma (note 8).

10 Thus, as the equality *a/b=(c/d)·(e/f)* means 'the ratio of *a* to *b* is formed by the ratio of *c* to *d* and by the ratio of *e* to *f*', it is necessary to have rules for the calculation of any one of the six numbers knowing the other five.

11 Respectively:

$$\frac{\text{Sin}\ \widehat{AE}}{\text{Sin}\ \widehat{EB}} = \frac{\text{Sin}\ \widehat{AW}}{\text{Sin}\ \widehat{WD}} \cdot \frac{\text{Sin}\ \widehat{GD}}{\text{Sin}\ \widehat{GB}} \quad \text{and} \quad \frac{\text{Sin}\ \widehat{AB}}{\text{Sin}\ \widehat{BE}} = \frac{\text{Sin}\ \widehat{AD}}{\text{Sin}\ \widehat{DW}} \cdot \frac{\text{Sin}\ \widehat{GW}}{\text{Sin}\ \widehat{GE}}$$

12 Thābit applies to the ratios AZ/EH, AZ/WT, WT/EH of Figure 15.3, where Z, T, H are the orthogonal projections of A, W, E on the plane GDB, a lemma established previously by similarity of right-angled triangles (Figure 15.2, Sin $\widehat{AE}/\text{Sin}\ \widehat{AZ} = EK/ZL$).

13 The theorem known as the 'rule of four quantities' appears in the same epoch. On this theorem and the sine theorem, see notes 15 and 21.

14 Al-Bīrūnī, *Maqālīd.*

15 Figure 15.8 later: Sin g/Sin *g*′=Sin a/Sin *a*′.

16 Literally 'the figure which dispenses' (from Menelaus's theorem), the term 'al-shakl' signifying both figure and theorem.

17 Compare Figure 15.5, from al-Nayrīzī, relative to the declination of the sun and where HN/ZH=BL/ZB leads to $\text{Sin}\ \delta_\odot/\text{Sin}\ \lambda_\odot = (\text{Sin}\ \varepsilon)/R$, with Figure 15.9 constructed by al-Khujandī who from HN/ZH=BL/ZB deduces

$$\frac{\text{Sin}\ \widehat{DE}}{\text{Sin}\ \widehat{AD}} = \frac{\text{Sin}\ \widehat{BG}}{\text{Sin}\ \widehat{AB}} = \frac{\text{Sin}\ \widehat{D'E}}{\text{Sin}\ \widehat{AD'}}$$

replacing D by another point D′ (see al-Bīrūnī, *Maqālīd,* pp. 148–9, 138–41).

18 Figure 15.8 later: Sin *b*/Sin *b*′=Tan *a*/Tan *a*′.

19 It is about the *Risāla fī al-Qusīy al-falakiyya* (Abū Naṣr, *Rasā'il*), translated and analysed by Luckey (1941). Elsewhere, most of the demonstrations of Abū Naṣr and of Abū al-Wafā' are reported in full by al-Bīrūnī (al-Bīrūnī, *Maqālīd,* pp. 110–37).

20 By convention 'Cos' is the Sine of the complement and $\delta_B(x)$ is the inclination of arc *x* for a maximum inclination equal to B.

21 Respectively: (Figure 15.7 taken from the *Risāla*)

$$\frac{\text{Sin}\ \widehat{AB}}{\text{Sin}\ \widehat{BG}} = \frac{\text{Sin G}}{\text{Sin A}}$$

i.e.

$$\frac{BM}{BL} = \frac{EH}{DS}$$

results from

$$\frac{BM}{BN} \cdot \frac{BN}{BL} = \frac{DZ}{DS} \cdot \frac{EH}{EZ} = \frac{R}{DS} \cdot \frac{EH}{R}$$

the perpendiculars to AGZ, BN and DS, coming together with BM and DZ when A is a right angle and (Figure 15.6 taken from the book of *Azimuths*)

$$\frac{\text{Sin}\ \widehat{DH}}{\text{Sin}\ \widehat{ZB}} = \frac{\text{Sin}\ \widehat{GH}}{\text{Sin}\ \widehat{GZ}} \qquad \text{by} \qquad \frac{HT}{ZL} = \frac{HK}{ZE}$$

22 Figure 15.9 relating to the tangent rule, corresponds to the other method, the same type as in the book of *Azimuths,*

$$\frac{\text{Sin}\ \widehat{AD}}{\text{Sin}\ \widehat{AB}} = \frac{\text{Tan}\ \widehat{DE}}{\text{Tan}\ \widehat{BG}} \qquad \text{translating} \qquad \frac{MD}{KB} = \frac{DY}{BH}$$

23 Latin transcription of Jābir.
24 The definition of these arcs is in Neugebauer (1975:61).
25 Cf. Neugebauer (1975:93–5, 183–4).
26 As in the *Almagest,* the arc obtained, $\Delta\alpha$, is the difference between the right ascension required, α, and the arc $\alpha^{-1}(\lambda)$ which is obtained immediately from the table of the right ascension of the sun $(\alpha\colon \lambda_{\odot} \mapsto \alpha_{\odot})$. It is impossible to give the details here of the procedure for these two formulae and the following two (cf. al-Bīrūnī, *Maqālīd,* pp. 55–7).
27 There are obviously several equivalent expressions for each of these four functions. Thus,

$$f_4(x) = \frac{\text{Sin}\ x.\ \text{Sin}\ \varepsilon}{\text{Sin}\ (90^\circ - x)} = \frac{R\ \text{Sin}\ \delta(x)}{\text{Sin}\ (90^\circ - x)}$$

by applying the formula mentioned above for the declination of the sun. With the conventions $\bar{x} = 90^\circ - x,\ \ \delta\colon \lambda_{\odot} \mapsto \delta_{\odot},\ \ \alpha\colon \lambda_{\odot} \mapsto \alpha_{\odot},$ it seems that the formulae at the origin of the construction of the table are

$$f_1\colon x \mapsto \delta[\alpha^{-1}(x)] \qquad f_2\colon x \mapsto \operatorname{Sin} \overline{\delta(x+90^\circ)}$$

$$f_3\colon x \mapsto \frac{R \operatorname{Sin} \bar{x}}{f_2(x)} \qquad f_4\colon x \mapsto \frac{R \operatorname{Sin} \delta(x)}{\operatorname{Sin} \bar{x}}$$

(cf. al-Bīrūnī, *Maqālīd,* pp. 56–7).

28 The two texts are edited in the collection Abū Naṣr, *Rasā'il.* See also Irani (1956).

29 Compare with Figure 15.10, in the *Almagest* of Abū al-Wafā', Figure 15.11 constructed by al-Bīrūnī where in their usual arrangement can be found the 'versed' or 'erect' shadow, KL, of the gnomon EK and the 'extended' shadow, ZT, of EZ, which is used to introduce the 'shadow' AS of arc AH and its 'extended shadow' BO.

30 Al-Bīrūnī, *Ifrād al-maqāl.*

31 The problem can be solved by Menelaus's theorem by constructing the measure of each of the required angles and appealing to a result in the *Almagest,* equivalent to the determination of two arcs knowing their sum or their difference and the ratio between their sines. Two methods of this type are applied by Abū al- Wafā' to the calculation of the hour angle, in a written piece obviously preceding his *Almagest.* The same principle can be found in the trigonometry treatises, with use of the triangle formulae.

32 Al-Bīrūnī introduces for this the configuration reproduced in Figure 15.12, which gives him pairs of triangles to which the rule of four quantities or the tangent rule is applied in order to solve the triangle ABG. Figures of the same kind will be used afterwards in order to establish the triangle formulae.

33 Hairetdinova (1966) and (1969).

34 Also edited in the collection Abū Naṣr, *Rasā'il.*

35 Figure 15.13, through

$$\frac{\mathrm{AE}}{\mathrm{AB}} = \frac{\operatorname{Sin} B}{\operatorname{Sin} E}$$

AE being equal to Sin *B* when AB=*R*=Sin *E*. Also,

$$\frac{\mathrm{AG}}{\mathrm{AE}} = \frac{\operatorname{Sin} E}{\operatorname{Sin} G}$$

whence

$$\frac{\mathrm{AG}}{\mathrm{AB}} = \frac{\operatorname{Sin} B}{\operatorname{Sin} G}$$

36 For instance, in the double-entry table which illustrates the study of the triangle, the ten classes according to sides and the ten classes according to angles are arranged with only a small exception, in the same order (first class, three acute angles, and

first class, three sides inferior to a quadrant, etc.). Naṣīr al-Dīn disregards the possibility of making the table symmetric by arranging the classes of one of the partitions in the order corresponding to the supplementary values of the classes of the other. In this context, it is clear that this symmetry would have been looked for if the connection had been made with the figure constructed for the three given angles.

37 Cf. Debarnot (1978).

38 Carathéodory (1891) (edited and translated from the text of Naṣīr al-Dīn).

39 For the three given sides, the first angle is calculated in the right-angled triangle determined by one side of the angle and its orthogonal projection on the other side, this being obtained by 'the ordinary rule' (*Elements* II, 12–13). The method is thus equivalent to the use of the plane theorem of the cosine.

40 Villuendas (1979) (edited and translated from the book of Ibn Mu'ādh).

41 Mu'ādh constructed (Figure 15.15) a point L such that $\widehat{LA} = x$ and $\widehat{LG} = y$ by $\widehat{AG} = \alpha$ and K such that GK/GA=TZ/TH=*b*/*(a−b)* following the trace of KLD. Taking into account that KO/OG=*(a+b)/(a−b),* the formula then results from BR/RD=KO/OD; cf. Villuendas (1979:11–21, 106–12).

42 As an example, Figure 15.16 reproduces one of the four figures constructed by Abū al-Wafā' for these two formulae (the case of the sum with $\widehat{AB}$, i.e. α, and $\widehat{BG}$, i.e. β, inferior to 90°):

$$TH = \frac{ZD}{2} = \frac{\text{Crd } (2\alpha + 2\beta)}{2} = \text{Sin } (\alpha + \beta)$$

The similarity of the triangles BNH and BTE leads to NH/BH=TE/BE, and the same for TN, whence the second formula, the first one going through $NH=(BH^2-BN^2)^{1/2}$ and BN/BH=BT/BE. In the *Almagest* the formula for the addition of chords is established with the help of a theorem said to be from Ptolemy.

43 The Arab authors indicate in this way the sides of some inscribed regular polygons, such as the square, the hexagon, etc., the knowledge of which allows the construction of the sine table.

44 Al-Bīrūnī, *Istikhrāj,* and Suter (1910a).

45 To five places, the calculation leads to

1; 2, 49, 48, 13<Crd 1°<1; 2, 49, 53, 4

(for Crd 1°=1; 2, 49, 51, 48).

46 King (forthcoming) *Spherical Astronomy in Medieval Islam:* The *Ḥākimī Zīj* of Ibn Yūnus, Frankfurt, ch. 10.

47 The coefficient 1/2 in the last calculation is replaced by 1/3, which will give Sin 1°=1; 2, 49, 43, 12. Putting Sin 2.1°−Sin(3°−1°)=*Rf(a),* with *a*=sin 1° considered as small, we can see in effect that $f'(a)$ is equivalent to 2+cos 3°, not much different from 3. It is intuitive that the error on Sin 2.1° is more or less twice that on Sin 1° or on Sin(3°−1°). However, the first calculation of Ibn Yūnus is not very exact: to five places, (8/9) Sin(9/8°)=1; 2, 49, 40, 8 and (16/15) Sin(15/16°)=1; 2, 49, 44, 34. It

needs then 1; 2, 49, 43, 5 instead of 1; 2, 49, 43, 28 for the first value.

48 Woepke (1860).

49 It should be 0; 31, 24, 55, 51, 57<Sin 1/2°<0; 31, 24, 55, 57, 37. Then, it is intuitive that the approximation would be better as Sin 1/2°=0; 31, 24, 55, 51, 57+(1/3)0; 0, 0, 0, 5, 40=0; 31, 24, 55, 53, 50 (to six places, Sin 1/2°=0; 31, 24, 55, 54, 0).

50 On the interval [15/32°, 18/32°], the procedure in the *Almagest* leads to

$$\frac{8}{9}\,\mathrm{Sin}\,\frac{9^\circ}{16} = 0;31,24,55,31,8 < \mathrm{Sin}\,\frac{1^\circ}{2} < 0;31,24,56,4,26 = \frac{16}{15}\,\mathrm{Sin}\,\frac{15^\circ}{32}$$

51 See in particular Hamadanizadeh (1979).

52 Al-Bīrūnī, *al-Qānūn al-Mas'ūdī*, book III, ch. 7 and 8. Translated by Schoy (1927).

53 On writing the formula

$$y = y_0 + \frac{x - x_0}{d}\left[\Delta y_0 + \left(\frac{x - x_0}{d} - 1\right)\Delta^2 y_{-1}\right]$$

we can see that a coefficient 1/2 is missing in front of the term

$$\left(\frac{x - x_0}{d} - 1\right)\Delta^2 y_{-1}$$

in order for the parabola by which al-Bīrūnī replaces the chord joining the coordinate points (x_0, y_0), (x_1, y_1) to go through a third point of the curve, here with co-ordinates (x_{-1}, y_{-1}). The error caused by this interpolation is still, approximately and to within a sign, equal to that resulting from linear interpolation.

54 Cf. Kennedy (1978) and Rashed (1991a).

55 These numbers, to be memorized, are respectively 39, 36, 31, 24, 15 and 5. They represent the first differences of the function $x \mapsto 150$ sin x for the values of x corresponding to half-signs of the zodiac, i.e. 150 Sin 15°, 150(sin 30°−sin 15°),…, 150(sin 90°−sin 75°) (cf. Brahmagupta, *The Khaṇḍakhādyaka*, ch. I, stanza 30, sine table; ch. IX, stanza 8, interpolation formula).

56 Cf. Hamadanizadeh (1979).

57 With standard notation, the formula of Ibn Yūnus is written in effect

$$y = y_0 + \frac{x - x_0}{2d}(\Delta y_0 + \Delta y_1) + 4\frac{x - x_0}{2d}\left(1 - \frac{x - x_0}{2d}\right)\left(\frac{\Delta y_0 - \Delta y_1}{2}\right)$$

i.e.

$$y = y_0 + \frac{\xi}{2}(\Delta y_0 + \Delta^2 y_0 + \Delta y_0) - 4\,\frac{\xi}{2}\left(1 - \frac{\xi}{2}\right)\frac{\Delta^2 y_0}{2}$$

whence the formula (see King, The *Ḥākimī Zīj,* ch. 10).

58 Al-Bīrūnī, *al-Qānūn al-Masʿūdi,* book III, ch. 3, and Schoy (1927).

59 As we have seen, these relations are not obtained from addition formula. The algebraic treatment *(bi-l-jabr wa-l-muqābala)* is only used to simplify the equations established geometrically, being respectively $(x^2-1)^2=x^2+x+1$ and $2x-x^3=1-x$.

60 Ḥabash takes $\theta=\theta_5$. The convergence is guaranteed by the fact that, m being equal to 24, the positive number $m\pi/180$ is strictly less than 1. Cf. Kennedy and Transue (1956) and also Kennedy (1969).

61 Notably Woepke (1954), and especially Aaboe (1954).

62 Cf. Sédillot (1853: pp. 77–83).

63 The reader can verify, after the calculation of q_k described in the following paragraph, that it is not necessarily thus (taking into account that $x<1$; 3, we only have $q_k \leqslant 64$) the eventuality $q_k>59$ corresponding to obtaining a preceding number less than unity from its real value when, in the equation giving q_{k-1}, x^3 is replaced by x^3_{k-2}. The error will correct itself at a later stage.

64 The method consists in calculating the side c_n of an inscribed regular polygon of $3\times2n$ sides from $c_n=(4R^2-u_n^2)^{1/2}$, where u_n, equal to Crd $(180°-120°/2n)$, is obtained from $u_0=R$ and $u_n=[R(2R+u_{n-1})]^{1/2}$. Al-Kāshī determines first the number n of dichotomies necessary to achieve the precision required. He finds $n=28$, $2\pi R=6$, 16; 59, 28, 1, 34, 51, 46, 14, 50 and then, after converting to a decimal fraction $2\pi=6.2831853071795865$; cf. Luckey (1950).

65 Hamadanizadeh (1980).

16
The influence of Arabic mathematics in the medieval West

ANDRÉ ALLARD

It is worthwhile to emphasize once more the important delay which exists between the knowledge Western medieval scholars had of Arab mathematical works and the works themselves. If one makes an exception of a single Latin manuscript which was attested to contain Indo-Arabic numerals from 976,[1] as well as the contributions of Gerbert of Aurillac and his successors in the area of abacus calculations, nothing appeared in the earlier Latin works before the twelfth century concerning the numerous Arabic works which had been developed during the period between the first quarter of the ninth century, the era of al-Khwārizmī, and the second quarter of the twelfth century, a little after the death of al-Khayyām (1123). Moreover, due to the Haskin's remarkable *Studies in the History of Medieval Science* we know that it was not until the second half of the eleventh century that the influence of Arabic sciences became evident in Latin works. Once more such influences only affected the works of Alfanus of Salerno, and more especially Constantine the African and his disciples Atto and Iohannes Afflacius who were concerned with the study of medicine.[2] However, these provide the first important evidence of an interest in oriental science which was to enjoy its greatest moments in the numerous translations of the twelfth century. Even if one could consider that Haskins has been entirely justified in using the term 'Renaissance' to describe this period, the fragmentary knowledge which scientific historians on the twelfth century possessed of several texts concerned with exact sciences merely allowed them to establish *status quaestionis* or to put forward several hypotheses which, even today, cannot be completely confirmed. A study of several of the most important twelfth century texts revealing an Arabic influence allows a more precise approach and a prudent revision of some opinions which were overhastily considered to be certainties. There remain too many rare Arabic texts written between the ninth and twelfth century which were made the subject of modern editions, particularly in the aforementioned area of arithmetic, for example the works quoted by Ibn al-Nadīm or al-Qifṭī, since when knowledge of the original sources of the first Latin translations has contained some serious deficiencies.

I do not propose here to describe in detail each medieval work which evinces Arabic influences, I will emphasize the first occasions, often unrecognized, of a lengthy western initiation period into arithmetic, geometry and algebra, as well as the most important later Latin works in these subject areas.

THE ARITHMETIC OF INDIAN RECKONING AND THE FIRST LATIN VERSIONS OF ARABIC ARITHMETIC

According to the testimony of William of Malmesbury, Gerbert of Aurillac (the Pope Silvestre II, died 1003) deserves the credit for having borrowed from the Arabs of Spain the device of the abacus made up of columns on which were placed counters *(apices)* marked or unmarked with figures.[3]

This is all the same as the 'Indian reckoning' *(al-ḥisāb al-hindī)*, a device using nine figures and a zero as the basis of mathematical operations, which in the twelfth century marked the first major transfer of Arabic science into the mathematical 'equipment' of Western science.

Muḥammad ibn Mūsā al-Khwārizmī, one of the members of the well-known 'House of Wisdom' in Baghdad, wrote towards 825, after his work on algebra, two arithmetical treatises, the originals of which have long since been lost.[4] Several twelfth-century texts contain different versions of this arithmetic in some twenty-four manuscripts:[5] *Dixit Algorizmi* (also sometimes known in accordance with Boncompagni as the *De Numero Indorum*); *Liber Ysagogarum Alchoarismi* (in four versions one of which is abridged); *Liber Alchorismi; Liber pulueris*. Independently from the links which unite these manuscripts,[6] one can suggest the summary given in Figure 16.1 of the relationships between the algorisms and the principal influences that are at work here.

A comparative analysis of these four texts, the oldest known algorisms together with those entitled *Helcep Sarracenicum,*[7] allows one to extricate the following principal elements in relation to their genesis and their Arabic sources.[8]

The *Dixit Algorizmi,* which has been edited several times[9] and contains several phrases which are not commonly used in Latin, but which reveal their Arabic origins, was considered for a long time to be the most ancient Latin version of al-Khwārizmī's lost Arabic text or even a translation of this text.

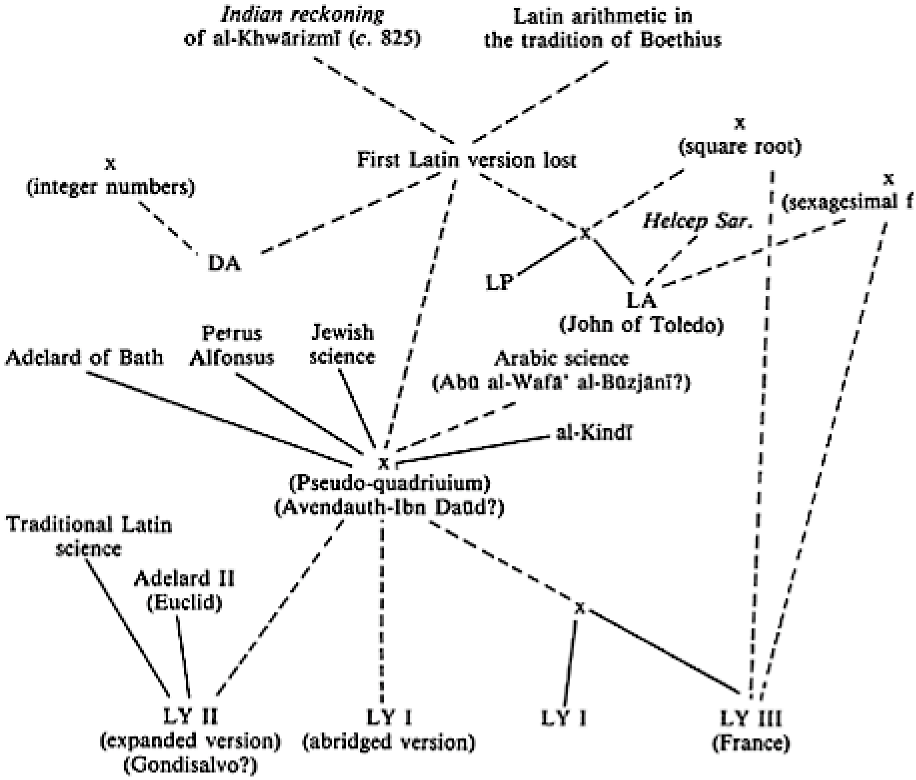


Figure 16.1

The examples which it contains are clearly separated into two distinct series, showing that it is in fact a hybrid text, certain elements of which have come from Boethius and Nicomachus of Gerasa. The work, compiled towards the middle of the twelfth century and known from a single incomplete manuscript, constitutes a precious piece of evidence, but not the sole one, of the birth of Indian reckoning in the West.

The arithmetical part of *Liber Ysagogarum Alchoarismi* makes up the first three tomes of five books, the last two of which are dedicated to astronomical and geometrical summaries which owe little to Arabic sources.[10] This work is hesitantly attributed according to Tannery and Haskins either to Adelard of Bath or Petrus Alfonsus. It was compiled a little after 1143 and is even more composite than *Dixit Algorizmi*. Within it one finds, according to the various versions, elements recognized as coming from Arabic works, such as those of Abū al-Wafā' al-Būzjānī or al-Kindī, but they also evince, at least in version one, a certain interest for the Jewish world and even for Hebrew. The 'Master A', cited by a single manuscript of version two, cannot be either Adelard of Bath or Petrus Alfonsus, but the influence of the latter is certain at least in the chapters concerned with geometry and astronomy. The *Helcep Sarracenicum* and the *Liber Ysagogarum* were written in a similar milieu to that of Adelard of Bath but this cannot be attributed to the English translator. A third version contained in a single manuscript was seemingly destined for France.

On the other hand, a whole passage of this same *Helcep Sarracenicum* can be found in the *Liber Alchorismi,* for which it constitutes a secondary source. *Liber Alchorismi* was almost certainly compiled in Toledo around 1143, and is by far the most elaborate work on the Indian reckoning prior to the *Liber abaci* of Leonardo Fibonacci (1202, revised 1228). But here again, at least in the second half of the work, there are some elements which are foreign to al-Khwārizmī. The author, a 'Master John', cannot be, as has often been suggested, the translator John of Seville, nor the friend of Plato of Tivoli, John David. In more than one aspect 'Master John' and 'Master A'of the second version of the *Liber Ysagogarum Alchoarismi,* appear to be one and the same. *Magister Iohannes* could be a member of the Chapter cathedral of Toledo, a collaborator, with Gundissalinus, of Avendauth, in all probability the Jewish philosopher Abraham ibn Daūd, who lived at Toledo between 1140 and 1180.[1]

The *Liber pulueris* contains long passages which are identical to those of *Liber Alchorismi,* but it also has many original passages. Although it was considered since its discovery to be a revision of *Liber Alchorismi, Liber pulueris* is in fact a more concise and probably older version inspired by the same Latin sources, the Indo-Arabic numerals which differ from those of its counterpart.

We will see later on several of the main elements inherited from al-Khwārizmī and other Arabic authors in Western Latin works. But one must note that the analyses of the numeral forms in the Latin manuscripts of the four major works which have been cited up to this point reveal several significant facts concerning the origin and the diffusion of the first Indo-Arabic numerals into the West.[12]

1 The differences between the figures is due to the evolution of the *ductus* of the Latin copyists which is linked to the method of writing left to right, whatever may otherwise be the eventual intervention in such evolution of the Visigothic script.[13]
2 Both *Dixit Algorizmi* and *Liber Alchorismi* contain proofs of the variety of ways of writing several numerals at the time, when the works were compiled. It seems to me that it is unlikely that this remark of the two authors is a result of the Indian reckoning of al-Khwārizmī.
3 The written forms which appear closest to the traditional Arabic series can be found in the hybrid version of *Liber Alchorismi* and *Liber pulueris* contained in two manuscripts. The difficulties of transcription to Latin writing from left to right are clearly illustrated here, particularly those concerning the number 3.
4 The manuscript 18927 from Munich, containing version III of the *Liber Ysagogarum,* explicitly defines the Toledian forms as distinct from the Indian ones.

In this way, despite the many differences and distancing from the primitive source, one notices that the West clearly kept a trace of the forms of the numerals close to those which it discovered in the twelfth century in the Arabic works concerned with arithmetic and astronomy, before the *ductus* of the Latin copyists gave them the appearance most commonly noted and a purely palaeographic evolution brought certain authors to recognize very early on, the existence of a variety of figures, of which they do not distinguish the common source. One can even note that, by differing degrees, the first Latin treatises of the Indian reckoning, particularly *Liber Alchorismi* and *Liber pulueris,* contributed to the diffusion of a technical vocabulary, the influence of which is evident in the best-selling works of the thirteenth century, for example those of John of Sacrobosco or Alexander of Villedieu. These texts, which were revolutionary novelties at the time, provided a happy result in clear contradiction to the recommendation made by the Muslim Andalusian author Ibn ʻAbdūn at the end of the eleventh century: 'One should not sell scientific books to Jews or Christians…since there will come a time when they will translate these and attribute them to their own people and their bishops, whilst they are in reality Muslim works.'[14]

THE LEGACY OF AL-KHWĀRIZMĪ AND OTHER ARABIC AUTHORS TO WESTERN ARITHMETIC

The elements which I have quoted show abundantly that one could search vainly through the twelfth century Latin texts claiming to find the legacy of al-Khwārizmī, an arithmetical tradition which would have remained unchanged during the three hundred odd years which separate these versions from the lost Arabic original.

The problem of the sources of the aforementioned Latin texts thus poses itself in a rather complex manner. It is aggravated even further if one considers that out with the *Dixit Algorizmi* (which once more cannot be regarded in an exclusive manner since one finds in the incomplete text traces of the Latin arithmetic in the style of Boethius), the references to al-Khwārizmī are infrequent. One cannot rest assured that in the twelfth century the words *alchorismus* or *alchoarismus* present in the title of the only version II manuscripts of *Liber Ysagogarum Alchoarismi* or the words *alchorismus, alghoarismus* or *algorismus* which are mentioned by John of Toledo referred again to the ninth-century Arabic author. These words doubtlessly designated the 'Indian reckoning' (*al-ḥisāb al-hindī*), i.e. the algorism based on the use of nine figures and a zero, in opposition to the traditional methods of the abacus and digital numeration. This second interpretation is certainly the one which must be retained for the title given to *Liber pulueris* in the hybrid version contained in the Palatin manuscript 1393 in the Vatican Library *(Incipit algorismus)*. Two passages allow us to shed light on this problem: the author of *Liber Alchorismi,* after having explained in detail and by various methods how to multiply $8\,\frac{1}{2}\,\frac{1}{4}\,\frac{1}{5}$ by $3\frac{1}{3}\,\frac{1}{9}$,[15] decided to multiply $8\frac{3}{11}$ by $3\frac{1}{2}$[16] at the same time stating explicitly that this was one of al-Khwārizmī's examples. Once again this quotation, if it really is one, is not strictly faithful because in the same circumstances both *Dixit Algorizmi, Liber Ysagogarum Alchoarismi* and even *Liber pulueris* multiply $3\frac{1}{2}$ by $8\frac{3}{11}$.[17] But another passage of *Liber Alchorismi* seems to indicate that the author is referring to an undefined authority.[18] Otherwise, however carefully one approaches certain passages of *Fihrist* and several authors who were writing treatises on the Indian reckoning after al-Khwārizmī and before the end of the twelfth century,[19] this primary observation is imposed. The examples used by the Latin texts to illustrate operations involving integer numbers are completely different from one another, except in some cases, but by no means all, where *Liber Alchorismi* and *Liber pulueris* have a common source, with the exception of several examples of the extraction of square roots[20] in those chapters which always remain distinct from the others concerned with fundamental operations, following on from the chapters relating to fractions. On the other hand, numerous examples of sexagesimal and ordinary fractions are used in all the texts. But neither these, nor the others, can be found in Arabic arithmetical texts edited today, which do not agree with each other any longer. Hence it is probable that al-Khwārizmī's original text did not contain examples but only, most likely, in a most succinct manner, a description of methods, at least for the more simple operations. One cannot exclude the possibility that the first lost Latin version included examples chosen by chance for the less ordinary operations (fractions and the extraction of square roots) which have been re-used by its successors. One is also led to conclude that only the methods explicitly described in the same manner by the Arabic and Latin authors can be considered as coming directly or indirectly from the first Arabic author who was inspired by Indian methods, although the order of presentation of these methods varies noticeably between the Arabic and Latin works. Amongst other operations, the multiplication of integer numbers was originally effected by a procedure which entailed the erasement of certain figures, as *Dixit Algorizmi* describes in the multiplication of 2326 by 214 which may be represented as shown in Figure 16.2.[21]

One can deduce that as well as including a study of Latin texts that the original Arabic work also contained a chapter on sexagesimal fractions[22] and another on ordinary fractions. It is possible that the two types of fractions were even partially mixed, because in the chapter dedicated to sexagesimal fractions one finds in both *Dixit Algorizmi, Liber Ysagogarum Alchoarismi, Liber Alchorismi* and *Liber pulueris,* the example of the multiplication of $1\frac{1}{2}$ by $1\frac{1}{2}$ by means of a reduction of sexagesimal fractions, to obtain the result 2° 15′ expressed as $2\frac{1}{4}$ by Dixit Algorizmi, Liber *Alchorismi* and *Liber pulueris,* but not by *Liber Ysagogarum Alchoarismi*. Conversely, each of these works taken in isolation offer their own characters which cannot be considered as coming from their ancient ancestor since the characters do not figure in the main body of evidence. Thus, *Liber Alchorismi* is the exponent of a system of fractions successively based on addition, by multiplying, for example, $8\ \frac{2}{3}\ \frac{3}{5}$ by $3\frac{1}{3}\ \frac{2}{4}$,[23] by analogy the expression of sexagesimal fractions as minutes, seconds, thirds, but also by a system of 'fractions of fractions' by multiplying, for example, $\frac{3}{8}\cdot\frac{1}{7}\cdot\frac{1}{10}$ by 4.[24] This mode of expression is absent from the other texts, especially *Liber pulueris,* and is mostly in evidence throughout the Middle Ages, as also within several Arabic works, mostly preceding the Latin algorisms[25] which should be considered as evidence of a tradition which was ill disposed to consider numbers other than units as denominators of fractions. Since when, an over

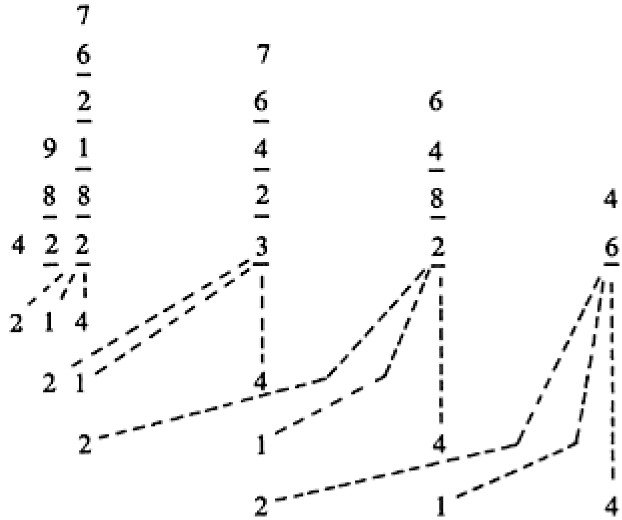


Figure 16.2

hasty appraisal of Latin arithmetical works has led to a rejection of the Arabic heritage of some chapters which are not recognized in Arabic works known today or an attribution of the methods described very exactly in the Latin texts to Arabic authors who came after al-Khwārizmī. On the contrary, it is my estimation that these chapters (and unfortunately the fragmentary state of the only manuscript containing *Dixit Algorizmi* does not permit us to study such items in this work) are worthy of attention. A clear example of the historical evidence which can be supported by the Latin texts is provided by the rule of approximation of the irrational square root, known as the 'rule of zeros' by Arabic authors, as it is described in detail by *Liber Ysagogarum Alchoarismi, Liber Alchorismi, Liber pulueris,* for example concerning the square root of 2.[26]

> The authors place an integer in opposition to a number pair of zeros, let us say six zeros. They then extract by the classical method of erasement of the root of 2 000000 to obtain 1414 'and a minimal remainder'. Then one considers that the units, tenths, hundredths of 1414 correspond to the half of the number of zeros attributed to the number and that the remaining unit is the integer from the square root of 2. One then reduces 0.414 into sexagesimal fractions in the following way: 414.60=24840, there are five positions, two more than half of the attributive zeros, from which the first root approaches 1° 24′; then 840.60, to obtain a final root which approaches 1° 24′ 50″ 24‴.

Liber Alchorismi and *Liber pulueris,* but not *Liber Ysagogarum Alchoarismi,* then state that instead of reducing to sexagesimal fractions, one can choose fractions of which the denominator may be 20, 30 or any other number, such as 2520 which has the advantage of being divisible by all the numbers from 1 to 10. Only *Liber Alchorismi* then defines its position on the problem of the expression of fractions from the approximate root in a manner surprising for its time.[27] Considering that the approximate root of 2 can be expressed just as well as 1 4/10 1/100 4/1000 without modifying the numbers obtained in the extraction and not only as 1° 24′ …, the author proved his comprehension of decimal fractions! The 'rule of zeros' described above is used by Arabic authors at the latest in the ninth century. Its general formula may be expressed thus:[28]

$$(a)^{1/n} = \frac{(a.10^{nk})^{1/n}}{10^k} \qquad \text{with } k = 1,2 \ldots$$

and the approximation thus obtained obviously includes the decimal fraction. The crux of the matter consists of determining how far the authors recognized the decimal representation of the fraction without being constrained to turn it into a sexagesimal fraction. In his exhaustive study of this subject, Rashed showed that the invention of decimal fractions should be attributed to the school of al-Karajī, particularly al-Samaw'al,[29] and not to such authors as al-Uqlīdisī (*c.* 952), nor to such Western authors as Stevin (1585) or Bonfils (1350). I believe that in concluding this analysis of the first twelfth-century Latin texts one can state that the 'rule of zeros' unanimously represented by *Liber Ysagogarum Alchoarismi, Liber Alchorismi* and *Liber pulueris* was already a part of the work of al-Khwārizmī, but the expression of the remainder must limit itself to an expression of sexagesimal fractions. The reflections purely expressed in *Liber Alchorismi* and absent from *Liber pulueris,* which in any case had an identical Latin source to the former, constrain us to attribute to John of Toledo the first Western manifestation of decimal fractions in his treatise of 1143. Was it an original creation or the representation of an earlier Arabic tradition, which if it did not explicitly define these fractions before al-Samaw'al, had at least approached them? In the absence of precise documentation we do not dare to resolve this question.

Probably and quite justifiably, one will not cease to mourn the loss of the al-Khwārizmī's works concerning arithmetic. Nevertheless, one can be sure that such works written by the same author as *Algebra* also constitute a basic source for a development which has been illustrated in the thirteenth century by works which were less elaborated than those of the middle of the twelfth century, such as the John of Sacrobosco's *Algorismus Vulgaris* and the Alexander of Villedieu's *Carmen de algorismo* (and also, although in a manner made less popular by the difficulty of access, by the Fibonacci's *Liber abaci*). The complexity of sources and the breaks in the actual knowledge concerning the transmission of the Arabic inheritance have had the effect that only a comparative and detailed analysis of the content of the ancient algorisms allows one to identify several processes and influences; the presence or the absence of one or another characteristic in the operations and the greater or lesser precision with which these operations are described allows us to situate a work, if not together with its sources, then at least in a movement in which other works responding to the same criteria meet.[30] This can be applied to the entirety of twelfth century and thirteenth century works, specifically dedicated to the study of the Indian reckoning of which the most well-known one:[31]

Dixit Algorizmi (first half of the thirteenth century)
  Liber Ysagogarum Alchoarismi (*c.* 1143)

Liber Alchorismi (*c.* 1143)
Liber pulueris (*c.* 1143)
Latin algorism of the abbey of Salem (twelfth century?)[32]
Latin algorism of the British Museum Royal 15 B IX (twelfth century?)[33]
Latin algorism of the British Museum Egerton 2261 (twelfth century?)
French algorism of the Bodleian Library Selden sup. 26 (thirteenth century?)[34]
Algorismus Vulgaris of John of Sacrobosco (thirteenth century)[35]
Carmen de algorismo of Alexander of Villedieu (thirteenth century)[36]
Ars algorismi, Bibliotheca Apost. Vatic. Palat. lat. 288 (thirteenth century)[37]

If one makes a systematic comparison of the methods described in these works[38] and in the Arabic treatises which are well known, such as the *Kitāb fī uṣūl ḥisāb al-hindī* of Kūshyār ibn Labbān (tenth-eleventh century)[39] or the *Kitāb al-fuṣūl fī al-ḥisāb al-hindī* of al-Uqlīdisī (tenth century),[40] one may put forward, for example, in relation to the subtraction of integers, a uniformly general method of conducting operations (such as layout of numbers, inscription of results, use of zero) which is quite remarkable. The most marked discrepancy is the way in which operations are initiated from the left or the right of the numbers. The most ancient Latin works, like those of the Arabs, describe only the most economical method from the beginning of the operation from the left, or at least showing their preference for this method *(Liber Alchorismi);* only *Liber Ysagogarum Alchoarismi* differs from this practice, but we know that its sources are particularly complex and thus its evidence concerning Arabic sources must be treated with caution. The rift is only produced in those towards the end of the twelfth century and the beginning of the thirteenth century which adopt almost exclusively the method of working from the right-hand number. It would seem that the proof by nine, often described in relation to multiplication, division, extraction of roots, does not figure in the older works and probably was not present in al-Khwārizmī's works either for addition or subtraction; it was very likely introduced by the latter at a later date by analogy.

A systematic comparison if thus carried out of all the Arabic works and their adaptations into Latin or Hebrew between the ninth and thirteenth centuries, concerning all the operations, would possibly, if it were ever carried out, allow one to gain a more clear picture of the Arabic evolution of Indian reckoning and of the benefits which the Latin West, confronted by multiple and often conciliatory traditions, did not fail to gain. The several quoted elements, which only constitute a first approach with hypotheses which are far too numerous, can only remain a modest outline.

The calculation procedures using nine figures and a zero and practising erasement of numbers in a 'dusty tablet' underwent its greatest diffusion due to two concise works at the beginning of the thirteenth century: *Algorismus Vulgaris* of John of Sacrobosco (John of Halifax) (*c.* 1256)[41] and the *Carmen de algorismo* of Alexander of Villedieu (Alexander of Villa Dei) (d. *c.* 1240).[42] These procedures were known to Fibonacci[43] but little recommended by him, they even resisted the use of paper and ink since one can find the texts described in detail and adapted anew in the commercial German arithmetic of Peter Bienewitz (Petrus Apianus) (1527);[44] several rare works of the twelfth century and the early thirteenth century, which I have already mentioned earlier in relation to subtraction, use these in a more exclusive manner. But it did not supplant the use of the abacus. This device, however, made certain operations, such as multiplication and division, very long and sometimes difficult. Other procedures known to Arabic authors were progressively imposed on the West, it would seem that Fibonacci with his *Liber Abaci* of 1202 was the likely initiator of this. The fact is particularly aptly illustrated by the example of multiplication.

In one follow-up to his *Liber Algorismi,* John of Toledo already showed proof of his knowledge of procedures which no longer used the erasement of figures, but rather by the addition of partial products, since one can read:[45]

$$23.64=4.3+10(6.3+4.2)+100(6.2)$$

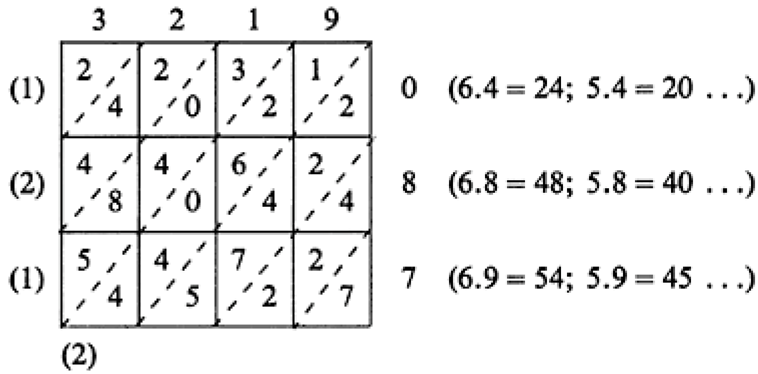
John of Sacrobosco operates the same way in his sixth rule of multiplication.[46] But the other authors limit this usage to numbers formed by units and tenths. The same method extended to some numbers figure for example in the arithmetic of al-Uqlīdisī (*c.* 952), under the name of 'method of the houses', as when multiplying 7254 by 4823 (the partial products written in squares of successive multiples of 10, starting on the right):[47]

…	48	23	12

i.e.

$$7254.4823=3.4+10(3.5+2.4)+100(3.2+8.4+2.5)\ldots$$
$$=12+10.23+100.48\ldots$$

This method is precisely that of the first proposed by Fibonacci in his *Liber Abaci* of 1202 to multiply, for example, 607 by 607;[48] one meets this under the influence of the former in the first anonymous Byzantine treatise on the subject of Indian reckoning in 1252[49] and in its successor written by Maxime Planude (*c.* 1292).[50] One also finds it in noticeably later works, Italian or German, for example *Arithmetic of Treviso* (1478), *Arithmetic of Bamberg* (1483), the works of Piero Borghi (1484), Francesco Pellos (1492), Luca Pacioli (1494) and also Niccolò Tartaglia (1556). But the 'method of the houses' was especially imposed under the form of a reserve where partial products were written which it was then necessary to add up diagonally to give them the true value of their position. Al-Uqlīdisī also operates in this way, for example the multiplication of 567 by 468, or 6583 by 489, in a manner which one may explain thus:[51]



(the addition of the numbers diagonally, starting with the bottom right square, and the writing down of the units gives the looked for answer 3219087)

Fibonacci called this method 'in the form of a chessboard' *(in forma scacherii)* and applied it for example, with an economical number of figures making some mental calculation necessary, to the multiplication of 4321 by 567.[52] Under the forms approaching or particularly in the manner known as 'blind' *(gelosia)* or 'mesh' *(graticola),* which did not differ from the former except by the inscription of all the numbers, one can see the same method employed by all the Western works renouncing operations by erasement, such as, to cite only the most well known, those of Nicholas Chuquet (1484), Luca Pacioli (1494), Niccolò Tartaglia (1556).[53] At the same time numerous Arabic authors such as Ibn al-Bannā' (d. 1321), al-Kāshī (d. 1429) and Bahā' al-Dīn (d. 1622) remained loyal to it.[54]

The example of multiplication which I have detailed suffices to suggest the influence which was exerted on medieval Western scholars by al-Khwārizmī and his successors. From the first Latin versions of the twelfth century until the more elaborate commercial Italian arithmetical works of the end of the Middle Ages and the Renaissance, the Indian reckoning which appears is that which was first elaborated by the Arabic authors, then transposed into Latin and finally into the vernacular. One cannot show completely today which texts of which authors, nor often even which contacts allowed the developments of the major stages, which we have emphasized, to occur, but we may rest assured that such is the case.

THE LEGACY OF ARABIC AUTHORS TO MEDIEVAL GEOMETRY

Earlier I made several allusions to the knowledge of the first Western authors on Indian reckoning of the most ancient Latin translations from an Arabic version of Euclid, particularly concerning the geometry section contained in the second version of *Quadrivium* which constitutes the *Liber Ysagogarum Alchorismi*. These allusions make it known that in this area the West was also indebted to Arabic authors for the discovery of true Euclidean geometry. The studies which have been made show that before the twelfth century only some rare Euclidean definitions were in circulation, compiled by the grammarians and showing the influence of several passages from the works of

Cassiodorus (d. *c.* 580) or of Isidore of Seville (d. 636). The sixth book of *De nuptiis philologiae et Mercurii* of Martianus Capella (fl. *c.* 470), although entitled *De Geometria,* is only a vague collection of these definitions, most of which are incomprehensible and which only contain one from amongst the most simple problems.[55] Other sources would have been more profitable if they had been better used. From the mathematical part of *Corpus* of the Roman *Agrimensores*[56] only some facts were in circulation as a summary measure like the area of a circle or the volume of a sphere, without even giving any knowledge of Pythagoras's theorem taken from these, whose applications are frequent and of which even then Franco of Liege (d. *c.* 1083) was ignorant.[57] Boethius (d. *c.* 524) most likely translated Euclid, or at least a part thereof, as Cassiodorus confirms.[58] But the work known as 'Geometry I'[59] and attributed to Boethius partially recovers the *Agrimensores* (Book I and *Altercatio* of Book V) or contains extracts from *Arithmetic* (Book II), and without the slightest demonstration gives the definitions, postulates, axioms and most of the propositions of the first four books of the *Elements* (Books III, IV and the beginning of Book V). The same is true of 'Geometry II', composed in Lotharingia in the first half of the eleventh century, based on a treatise on the abacus by Gerbert (d. 1003), the *Agrimensores* and extracts from Euclid which resemble those of 'Geometry I'.[60]

Before the Renaissance of the twelfth century, the influences of the Euclidean survey were limited in the West to a practical geometry and summary. It is from this point of view that one must consider Gerbert's school at Reims, or that of his disciple Fulbert (*c.* 965–1028) at Chartres. The reason for this impoverishment of the upper Middle Ages can be found quite simply in the almost total absence of scientific texts; such penury confined authors to the limits of calculus, where they did of course excel, but which left them strangers to demonstrative reasoning.[61] The discovery by Western Latinists in the twelfth century of the Arabic translations of Euclid was thus a departure point for a scientific revolution. From 1880 Weissenborn brought attention to bear on a Latin translation of the *Elements* by Adelard of Bath,[62] which had until then eclipsed the famous and widely used commentary of Campanus of Novara (*c.* 1255), and Björnbo had done the same with a Latin translation of Gerard of Cremona, discovered by him in 1901.[63] However, the first important glimmer concerning the first evidence of a rediscovery of Euclid was given by Clagett in 1953,[64] then by Murdoch in 1968.[65] Important work still in progress tends since then to give a clear picture of the numerous Euclidean texts of the twelfth and thirteenth centuries:[66] I will summarize the principal conclusions of these works below.

From the starting point of several Greek manuscripts obtained during the Byzantine Empire several Arabic translations were made of the *Elements* of Euclid. Al-Ḥajjāj (*c.* 786–853) completed two, the first of which was lost, and the second, a much shorter version written during the time of the Caliph al-Ma'mūn, included comments by al-Nayrīzī (d. *c.* 922). Isḥāq ibn Ḥunayn (d. 910) made another translation which is only known now in the revision of Thābit ibn Qurra (d. 901); the non-Euclidean books XIV and XV were translated in Baghdad by Qusṭā ibn Lūqā (d. *c.* 912).[67] Certain parts of these texts allow one to believe that the translations are linked and in particular that certain manuscripts revised by Thābit ibn Qurra take up the tradition of al-Ḥajjāj, especially with regard to the arithmetical part of the *Elements* (books VII to X).[68]

The medieval West made the greatest gains from these translations of the *Elements*. Three Latin versions of the Arabic Euclid are usually attributed to Adelard of Bath (*c.* 1080–1150),[69] as well as the version of *Liber Ysagogarum* erroneously attributed to the same author.[70] Another version is probably due to Herman of Carinthia (fl. *c.* 1140–50)[71] and the prolific translator Gerard of Cremona also completed another.[72] The version known as Adelard I is a particularly close translation, for the most part, of the revision by Thābit ibn Qurra, or through this of the translation by Isḥāq ibn Ḥunayn, but some passages are closer to the tradition of al-Ḥajjāj.[73] It is thus a hybrid version composed most probably during the second quarter of the twelfth century, which does not seem to have been written by Adelard himself and which contains the non-Euclidean books XIV and XV, but not book IX and propositions 1 to 35 of book X of the *Elements*. The version often called Adelard II, which seems to be the work of Robert of Chester, was very successful in the Middle Ages, but its history is particularly complex; from what we know today it appears that it underwent some revision.[74] Although he is probably the author, Adelard himself is actually quoted in it: this fact is not exceptional in medieval works. Boethius or his source Nicomachus and Cicero[75] are numbered amongst the sources, as well as an 'Eggebericus' and a 'Reginerus' which we have not been able to identify,[76] an Ocreatus (or Ocrea) who could be Nicholas Ocreatus, a pupil of Adelard to whom he dedicated his treatise on arithmetic,[77] and a Robertus of Marisco who could be Robert Marsh, parent of Robert Grosseteste (d. 1253) and archdeacon of Oxford.[78] In a form which is perhaps even older than that revealed today by the manuscripts, this second version was undeniably translated from Arabic, although a Greco-Latin influence was also present.[79] A third version differing greatly from the first took definitions, postulates, axioms and statements of propositions from the second and added numerous demonstrations which these inspired. This third version was known to Roger Bacon (*c.* 1214–92) as *editio specialis Alardi Bathoniensis*.[80] Although it contained Arabic terms which were foreign to the second version, it seems to be more a commentary than an independent translation.

The translation attributed to Herman of Carinthia, known by a single manuscript which is missing books XIII to XV of the *Elements,* seems to have been much less successful than the preceding ones. Recent studies concentrating essentially on the texts of definitions have shown that there are undeniable links between the first two Adelardian versions, the translation of Herman of Carinthia and certain passages of the enlarged version of the *Liber Ysagogarum*. It would seem that version II of Adelard occupies an intermediate position between version I and that of Herman and that certain passages were redone in the enlarged version of *Liber Ysagogarum*.[81] The editor has shown that Herman's text as we know it today is noticeably abridged and has a different aura in the hybrid edition contained in manuscript *Reginensis* 1268 in the Vatican Library.[82]

The vagaries of the transmission of texts and their distribution have meant that the translation of the *Elements* accomplished by Gerard of Cremona,[83] a great translator of the twelfth century, did not enjoy as much success as the second Adelardian version; however, this translation constitutes the most complete version of the *Elements* known to the Latin West before the rediscovery of the Greek text. This is not surprising when one realizes that, more faithful to the traditions of Isḥāq ibn Ḥunayn and Thābit ibn Qurra than that of al-Ḥajjāj, it includes numerous Euclidean elements which are absent from the other texts cited:[84] the very quality of the principal source, very faithful to the original Greek, justifies the superiority of this Latin translation. Gerard of Cremona was also the author of a translation of the commentary of al-Nayrīzī on the first ten books of the *Elements,*[85] a commentary of book X by Muhammad ibn ʿAbd al-Bāqī[86] and a fragment of the commentary of Pappus of Alexandria on book X, translated by al-Dimashqī.[87]

It was not inevitable that the Arabs would be the intermediary to bring the knowledge of Euclid to the Latin West. Towards 1160, books I to XIII, book XV and a résumé of books XIV and XV of the *Elements* were translated from Greek into Latin by an anonymous student in Sicily, probably the same one who, coming from Salerno, translated the *Almagest* of Ptolemy.[88] But the influence of his work was in no way comparable with the earlier and contemporaneous translations of Euclid from Arabic, which supplanted the treatises of practical geometry inspired by the *Agrimensores* and the Arabic treatises concerning the use of the astrolabe, such as the *Practica geometriae* of Hugh of St Victor (*c.* 1086–1141).[89] It is certain that no commentary worthy of interest was compiled before the middle of the thirteenth century and that that of Albert the Great relied heavily on the commentary of al-Nayrīzī.[90] But a systematic study of the very numerous Latin manuscripts, such as is being undertaken today, particularly concerning the manuscripts of the thirteenth and fourteenth centuries, denotes a prodigious explosion of interest in the new vision of Greek science which the Arabic translations of Euclid had introduced to the Latin West in the first half of the twelfth century. One example amidst dozens of others allows us to illustrate this.[91]

On folio 49r of Latin manuscript 73 in Bonn University Library (thirteenth century) and on folio 38r of the Latin *Reginensis* 1268 in the Vatican Library (fourteenth century), at the end of book VIII of the *Elements,* can be found a rule of proportions:

> For three given quantities the ratio of the first to the third is equal to the product of the ratio between the first and the second and the second and the third.[92]

There follows a demonstration which can be explained thus.

Let

$$\frac{b}{a} = d \qquad \frac{c}{b} = e \qquad d.e = f$$

Because

$d.a=b \qquad d.e=f$

one obtains

$$\frac{e}{a}=\frac{f}{b} \qquad e.b=f.a \ (\textit{Elements VII,19})$$

Now

$e.b=c$

whence

$$f.a=c$$
$$\frac{c}{a}=f=\left(\frac{b}{a}\right)\left(\frac{c}{b}\right)$$

This demonstration corresponds in another form to the commentary (II, 4) of Eutocius on *The Sphere and the Cylinder* of Archimedes.[93] Expressed geometrically the rule is the fifth definition of book VI of the *Elements* in the Sicilian translation of the Greek text[94] and with this slight exception was known as such in the Latin West through the translation of the Arabic text by Gerard of Cremona.[95] It can be found, without proof, in a translation again by Gerard of Cremona of *Epistola de proportione et proportionalitate* by Aḥmad ibn Yūsuf (d. *c.* 912)[96] cited by Campanus of Novara (d. 1296), Leonardo Fibonacci (1202)[97] and Thomas Bradwardine (d. 1349).[98] The demonstration also appears in the anonymous *Liber de proportionibus* attributed to Jordanus Nemorarius (d. 1237) and in the *Tractatus of proportione et proportionalitate* attributed to Campanus of Novara,[99] as well as (IV, 27) in the *Liber de triangulis* of PseudoNemorarius[100] and in the notes of Roger Bacon (d. *c.* 1292) on the *Elements*.[101] Another demonstration resembling that of Eutocius can be found in the treatise on optics *(Perspectiua)* (*c.* 1270) of Witelo.[102] In the fourteenth century, the statement and the demonstration could be found in such works as *Quadripartitum* by Richard of Wallingford (d. 1335), one of the first continuations in Latin of a work on trigonometry by al-Ṭūsī,[103] and in the *De proportionibus uelocitatum in motibus* of Simon of Castello, attributed to Nicholas Oresme;[104] only the statement is found in the *Tractatus de proportionibus* of Thomas Bradwardine (1328)[105] and in the *Tractatus proportionum* of Albert of Saxony (*c.* 1316–90).[106] Analogous research in later works does not fail to reveal the same imprint.

We have alluded to the commentaries of the *Elements* made by Albert the Great and Roger Bacon, based on the second and third versions of Adelard; both draw heavily on the commentary by al-Nayrīzī translated by Gerard of Cremona.[107] But of all the works inspired by the Arabic version of Euclid, the *Commentary*[108] of Campanus of Novara, which in fact consists of the ‘first edition’ of Euclid (Venice, 1482) compiled very likely between 1255 and 1261, is clearly the one whose influence on Western science and its diffusion has been most determinative: the very high number of manuscripts of this work and the thirteen or so successive editions in the fifteenth and sixteenth centuries alone bears witness to the success of this work beyond contest. However, owing to the lack of an exhaustive study of this subject, we still have only patchy knowledge of the diverse sources of Campanus. Amongst these are certainly version II of Adelard of Bath, the commentary of al-Nayrīzī (Anaritius),[109] the *Epistola* of Aḥmad ibn Yūsuf, mentioned several times by the author under the name Ametus filius Josephi,[110] the *Arithmetic* of Nemorarius and the *De triangulis* of Pseudo-Nemorarius.[111] We cannot mention here the medieval works in which the Euclid of Campanus is the determining factor in the evolution of scientific thought: the influence of the rediscovery of Euclid through the intermediary of translations and the original Arabic works extends beyond the framework of scientific literature to become the basis of the teaching for all sciences and for all encyclopaedic knowledge.[112] It would be symptomatic of this point of view to emphasize the difference in quality which separates the *Practica geometriae* of Hugh of St Victor, compiled only from knowledge of Boethius, from the *Agrimensores,* the *Quadratum geometricum* of Gerbert and from Arabic works on the astrolabe, even from the *Practica geometriae* of Fibonacci (1220) or of Dominicus de Clavasio (fl. *c.* 1346), for example, where the influence of the Latin translations of the Arabic Euclid is always present.[113] Although the contribution to scientific progress in the Latin West represented by knowledge of the *Elements* of Euclid was fundamental, it was not exclusive. Whatever the state of ignorance in which we find ourselves with regard to the true sources of Leonardo Fibonacci's work,[114] certain facts are disturbing. Thus, the fourth part of *Practica geometriae* (1220), entitled *De divisione omnium camporum inter consortes* ('On the division of all fields between the presumptive inheritors'), is the first Western reflection on the lost work of Euclid on the division of geometrical figures,[115] cited by Proclus in his commentary on the first book of the *Elements*. The work is a revision used by several authors[116] and adds to the propositions some numerical examples which justify the title, but no less than twenty-two of the propositions which it contains are treated in a manner which is more or less identical with that used in an Arabic text,[117] eight other propositions from the latter are clearly cited by Fibonacci and the six last are considered as known in the demonstrations.[118]

We cannot emphasize enough the major influence of the Arabic version of Euclid and its diffusion in numerous medieval works. Other works, and not the lesser ones, were known in the West through the intermediary of Latin translations realized by Gerard of Cremona. Thus as a result of the masterly effort dedicated by Clagett to the Arabo-Latin traditions of Archimedes[119] we know how the work of the Greek mathematician was revealed. Despite the considerable contribution of the translations of the Greek text made by a friend of Saint Thomas of Aquinas, William of Moerbeke (*c.* 1215–86), the influence of the Arabic version of Archimedes went much beyond the bounds of the twelfth and thirteenth centuries. To be convinced, it suffices only to note that a work such as the *Liber de motu* of Gerard of Brussels (thirteenth century), however dependent according to its author of his own authority, is intimately linked to *De mensura circuit* translated by Gerard of Cremona.[120] It is perhaps the same with the *Liber de curuis superficiebus* of Johannes de Tinemue (John of Tynemouth?) (d. *c.* 1221):[121] together with the *De mensura circuit* this constitutes in the thirteenth and fourteenth centuries the most popular of the works inspired by Archimedes, contributing with the *Verba filiorum*

Moysi of the Banū Mūsā to making known in the West the propositions of the first book of *De sphaera et cylindro* and used for example by Nicholas Oresme (*c.* 1320–82), Francis of Ferrara (1352), the anonymous author of a commentary on *Liber de ponderibus* and the anonymous author of *Liber de inquisicione capacitatis figurarum* (fourteenth to fifteenth century).

A systematic account of Arabic influence on the use of Archimedes' work in medieval science would not forget at least to mention, even if we do not repeat it here, the work of Jordanus Nemorarius, Leonardo Fibonacci, Roger Bacon, Campanus of Novara, Thomas Bradwardine, Francis of Ferrara, Nicholas Oresme, Albert of Saxony, Wigandus Durnheimer and numerous anonymous authors. In the actual state of knowledge it is often difficult to identify what is due to Arabic influence and what comes from the Greek text or the thirteenth-century Latin translation by William of Moerbeke. But certain facts should be emphasized. Thus one can pick them out, for example, in the Latin solutions to the famous problem of the trisection of an angle.

Apart from this particular case of a secant from the edge of a diameter perpendicular to any chord, the problem of a secant from a point and intercepted on a given length by two straight lines does not provide solutions by means of a ruler nor of a compass since it leads to a search for the points of intersection of the hyperbola $y(c-x)=ab$ and the parabola $x^2=ay$.[122] This secant is used by Archimedes in his propositions V to VIII in the *On Spiral lines* and in the eighth proposition of the *Lemmas (Liber assumptorum)* which are only known to us through an Arabic revision.[123] In his classical work on Greek mathematics Heath showed that the problem of the secant is linked to that of 'inclinations' (νεύσεις), evoked by Pappus, and to the trisection of an angle,[124] but we do not know how Archimedes resolved the problem of the secant. This problem, like that of the trisection of an angle, was revealed to the West through the Latin translation by Gerard of Cremona of the *Book of the knowledge of measurements of plane and spherical figures* by the three sons of Mūsā ibn Shākir, known under the title *Liber trium fratrum*[125] and most frequently as *Verba filiorum Moysi*.[126] The eighteenth proposition of *Verba* proposes a solution to the problem of the trisection of an angle which can be stated as follows (Figure 16.3).[127]

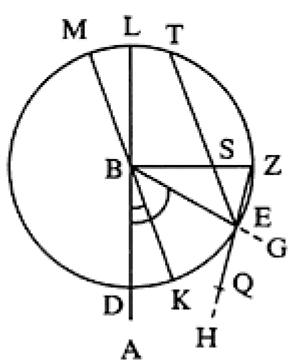


Figure 16.3

The trisection of the acute angle ABG is obtained through the 'inclination' in the direction of L of the chord ZE extended to ZH (a chord obtained by joining point Z, the end of radius BZ perpendicular to the straight line LA, and point E, the intersection of the straight line BG with the circumference of radius BD), by keeping point Z at the circumference and point E at the intersection of BG and the circumference until segment ZQ, equal to the radius of the circumference, is equal to the segment TS on the secant TE realized by the 'inclination'. The angle DBK obtained by drawing MK parallel to TE and passing through the centre B of the circumference is the required third of the angle ABG.

This mechanical solution is of a similar nature to that of the drawing of a conchoid of a circle used for the same purpose by Roberval.[128] In many fine details this solution corresponds to the first of the solutions to the same problem provided by the anonymous *Liber de triangulis* inspired by the *Liber philotegni* of Jordanus Nemorarius.[129] In a second solution, a variation on the first, the author chose to 'incline' the straight line LN by displacing point L on the circumference in the direction of Z and maintaining point E at the intersection of the circumference and the straight line BG until the segment LO equals the radius of the circumference attained by radius BZ; thus he obtains the same secant TSE as the first solution (Figure 16.4).

But the text clearly indicates that neither of the mechanical solutions really satisfies the author.[130] He prefers a geometrical solution which consists of directly creating the secant TSE in which the segment TS equals the radius of the circumference, citing on this occasion proposition V, 19 of a *Perspectiva*. In this construction based on conical sections the editor has demonstrated the influence of the *Optics* of Ibn al-Haytham (Alhazen) conforming to the tradition of the text as revealed by the manuscripts of the Royal

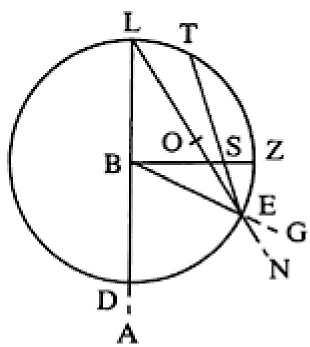


Figure 16.4

College of Physicians in London.[131] One cannot be surprised at the fact in so far as Ibn al-Haytham was the author, after Ibn Sahl and others, of a reform of optical geometry. We especially stress here that recourse is again made to an Arabic work and its translation. Inserted into the 'first edition' of the *Elements* of Euclid (Venice, 1482), stemming from the commentary of Campanus of Novara, the third solution of the author of *De triangulis* became an integral part of the teaching of geometry,[132] without mention of the *Perspectiva*.

The influence of the *Verba filiorum* by the Banū Mūsā is not limited to the work of the author of *De triangulis* nor to that of Roger Bacon. This influence is as palpable, for example, in the geometrical part of Latin manuscript 7377B, in the Bibliothèque Nationale of Paris (fourteenth century), with regard to the area of a circle or a triangle, in the Pseudo-Bradwardine (fourteenth century) or in the *De inquisicione capacitatis figurarum* (fourteenth to fifteenth century). And above all, the textual similarities between the *Verba filiorum* and the *Practica geometriae* of Fibonacci (1220) with regard to the area of a circle, the Heronian formula for the area of a triangle, the area of a cone or a sphere and the search for two means continually proportional to two given quantities clearly indicates the sources of the great mathematician of Pisa. It has also been put forward, for example, that the Heronian formula for the area of a triangle as a function of its sides[133] appears in such works as *Artis metrice practice compilatio* by Leonard of Cremona (*c.* 1405), without demonstration, in the *Summa* of Luca Pacioli (1494), with a demonstration taken from Fibonacci, in the German commercial arithmetic of Johannes Widmann; it was used by Pierre de la Ramée (Petrus Ramus) (1589) with an identical demonstration to that of the *Verba filiorum,* and certainly also by many authors of the sixteenth century.

The brief considerations above, where we chose deliberately to illustrate the progress of medieval science through the discovery of Arabic translations of Euclid and Archimedes, could lead to the conclusion that the West was only concerned with these Arabic works, even before the Renaissance, in order to renew by means of the scientific inheritance of Greece the poor remainders of the legacy of Boethius geometry. This would be a grave travesty of the work of the twelfth-century translators, the importance and the diffusion of which we have only implied. Although the most important and well-known translator, Gerard of Cremona, made a real contribution to the spread of knowledge, at least partially and for us limited to the domain of geometry, about the works of Euclid, Theodosius, Archimedes, Menelaus and Diocles, much more numerous are the Arabic authors, compilers, translators, commentators, not to mention the original thinkers, from whom medieval scholars learnt a great deal through Latin translations: the Banū Mūsā, Aḥmad ibn Yūsuf, Thābit ibn Qurra, Ibn ʿAbd al-Bāqī, Abū Bakr al-Ḥasan, al-Nayrīzī, al-Kindī, to cite, in arbitrary order, only those authors whose works had a direct influence on geometry and were translated by Gerard of Cremona. In this context, where we have emphasized several important points, it is appropriate to include works such as the *Liber de speculis comburentibus* and the *Liber de aspectibus* (or *Perspectiva*) of Alhazen: Gerard of Cremona was definitely the author of a translation of the first of these works and perhaps of the second, which brought conical sections to the attention of the medieval West. These two works were completed by the translation of the *Liber de duabus lineis,* towards 1225, by John of Palermo, a member of the Sicilian court of Frederick II of Hohenstaufen, then by the translations of Archimedes and Eutocius compiled by William of Moerbeke (1269) and at the end of the thirteenth century by an anonymous treatise *Speculi Almukefi compositio*. The importance of these texts and their links with works such as those of Witelo (1270), John Fusoris (1365–1436), his contemporary Giovanni Fontana and John Müller (Regiomontanus) (1436–76) does not need to be repeated, and neither does their influence on fourteenth-century works.[134] The reading broached with enthusiasm in the twelfth century was retained until the more advanced periods of Western science, which despite its lack of knowledge of the fact is no less a beneficiary. However, one must not forget that the medieval interest in geometry, at first limited to a summary approach inherited from Boethius in the form of the *Quadriuium,* later remained intimately linked to the study of philosophy and not to mathematical science as its real subject. One can thus realize that the important reflections of Arabic mathematicians concerning the postulate of Euclid on parallels found no echo in the medieval Latin world.[135]

THE EARLY STAGES OF ALGEBRA AND THE INFLUENCE OF ARABIC SCIENCE

In describing the main outlines of the Arabic legacy in the fields of arithmetic and medieval geometry, as we have attempted to do above, we have merely emphasized the long continuity of Western teaching, whose roots spring from the Latin translations of Arabic works in the course of the Renaissance of the twelfth century.[136] In the area of algebra, original research which has brought fame to the greatest names of Western science since the beginning of the modern epoch, the limited interest of historians of science in medieval sources, the slow and often recent discovery of original Arabic works greatly superior to their Latin contemporaries of the Middle Ages have relegated to second place the most ancient evidences, often only the subject of partial studies.[137] Thus it is often recalled without other comment that the very term algebra came from the work of al-Khwārizmī; in the same way one mentions that the first manifestation in the West of the influence of the brilliant Greek precursor Diophantus of Alexandria can be found in the work of Leonardo Fibonacci from 1202: an exact affirmation, it is true, but misleading in so far as it omits to be specific about a necessary Arabic intermediary.[138] Thus one should not be surprised if we endeavour to clarify here the content of the ancient Latin works, to the modest extent that they reveal their sources.

A little before the middle of the twelfth century, the West discovered how, by *al-jabr,* one can solve an equation of the second degree reduced to canonical form (i.e. by reducing the first coefficient to one) by only conserving the positive terms in the two members by adding the same amount, and how one reduces similar terms by *al-muqābala*. This proposition constitutes the first part of the famous *Concise Book of Algebra and al-muqābala*[139] written by al-Khwārizmī, which by good luck is available in the Arabic text, in contrast with the two arithmetical works written by the same author. We have shown that it is highly likely that the *Liber Alchorismi de pratica arismetice,* the most detailed and complete of all the ancient works stemming from the arithmetic of al-Khwārizmī, was compiled by a ‘Magister Iohannes’, John of Toledo, a collaborator of Avendauth, and not by the well-known Latin translator John of Seville (Iohannes Hispalensis) as is indicated by the sole very corrupted manuscript 7359 in Paris. The extensive considerations without title, which follow the Indian reckoning in the same work, are less well known.[140] Here one can find reflections on integers, fractions and proportions, issues of traditional Latin arithmetic, several practical arithmetical problems and even, but again only in manuscript 7359 in Paris, a magic square.[141] The definitions themselves show that the author used the preceding Indian reckoning,[142] but in particular one may find under the title *Exceptiones de libro qui dicitur gebla et mucabala*[143] a short chapter which describes al-Khwārizmī's three trinomial equations reduced to their canonical form,[144] followed by numerical applications.

Since 1915 it has been known that Robert of Chester (Robert of Ketene) completed a translation of the *Algebra* of al-Khwārizmī,[145] very likely in 1145, a little after having momentarily abandoned the scientific domain to realize the first Latin translation of the Koran (1141–3) at the behest of Peter the Venerable. It is difficult to give him unreserved credit for the version of the text edited under his name and stemming almost exclusively from a manuscript copied by the German mathematician Johann Scheubel; this is a revision written by Scheubel himself. He added several calculations to the text and replaced certain original terms by others better known at the time (*census* for *substantial*); several paragraphs which appeared neither in other Latin versions nor in the Arabic text can only be attributed to Scheubel.[146] However, from the remarks of Björnbo,[147] it has been agreed to recognize Gerard of Cremona as the author of the third version;[148] a version attributed to the same translator and published in 1851[149] is recognized as a revision: the preferred text appears to have been translated from Arabic, unlike its successor.

If one accepts the hypothesis that the *Liber Alchorismi* of John of Toledo is a homogeneous whole in which Indian reckoning occupies the most important position, the fragment of algebra is without doubt contemporaneous with the translation of Robert of Chester and represents with it the first Latin manifestation of the work of al-Khwārizmī, largely supplanted a little later by the translation of Gerard of Cremona. In the absence of an exhaustive study of these three versions and their relationships with the Arabic text, we can only make the point that, despite its brevity, version I moves away noticeably from the Arabic text and versions II and III.[150] In the example accompanying the second canonical equation $(x^2+q=px)$, one notices that in versions II and III the equation is solved by the expression

$$x=\frac{p}{2}\pm\left[\left(\frac{p}{2}\right)^2-q\right]^{1/2} \qquad \text{if } \left(\frac{p}{2}\right)^2>q$$

This expression is applied to an example chosen from versions II and III and also from the Arabic text ($x^2+21=10x$) and results in the two roots $x=3$ and $x=7$. Only version I provides, without commentary, an example with a single root ($x^2+9=6x$, in which $(p/2)^2=q$) which can be found in the *Algebra* of Ibn Turk, a contemporary of al-Khwārizmī, but not in the work of al-Khwārizmī, although it does in fact correspond to the general case envisaged by al-Khwārizmī's Arabic text (which this author declared to be impossible when $(p/2)^2<q$): 'thus the root of the square is equal to the half of the roots, exactly, without any extra or any short'. This general case is translated through versions II and III;[151] Fibonacci defined the same notion in 1202 in his own terms.[152]

In the period following the first Latin translations, the lesson of al-Khwārizmī's *Algebra* had differing effects and was selectively retained. Thus Jordanus Nemorarius in the *De numeris datis* (beginning of the thirteenth century) (propositions IV, 8, 9, 10) explains in his own way with his own examples the three trinomial equations reduced to their canonical form.[153] In the *Liber abaci* (1202) Fibonacci takes up the complete explanation of the three binomial equations plus the three trinomial equations together with Arabic demonstrations through the equality of areas[154] and numerous examples, occasionally original; even the title, *secundum modum algebre et almuchabale,* clearly indicates the source.[155] Following these two authors who constituted to differing degrees the basis of the Western apprenticeship in algebra, all the authors of the Middle Ages and the Renaissance, whom it would be impossible to cite here, continually reproduce the same proposition, sometimes with subtle subdivisions which reached an extreme with Piero della Francesca (*c.* 1419–92) of sixty-one classes of equations.[156]

One might be surprised that neither the second part of al-Khwārizmī's *Algebra,* dedicated to the calculation of surfaces for surveying, nor the third part, dedicated to problems relating to successions or wills and dealing by chance with several problems of Diophantine analysis, were translated in the twelfth century. The Arabic text which the Latin translators used was perhaps only concerned with algebra; moreover we have seen that one of the first translators, John of Toledo, had only a misconceived view of al-Khwārizmī's work. However, in 1145, the year when Robert of Chester probably completed the first Latin translation of *Algebra,* Plato of Tivoli translated a work written in Hebrew in 1116, the *Liber embadorum* of Abraham bar Ḥiyya (Savasorda) whose sources, we know today, were at least partially Arabic and consisted of a form developed in the second part of al-Khwārizmī's work.[157] A work of the same nature attributed to a poorly identified Arabic author, Abū Bakr, was translated in the third quarter of the twelfth century by Gerard of Cremona under the title *Liber mensurationum*.[158] The translation of a fourth work, the *Algebra* of Abū Kāmil (*c.* 850–930), brought to medieval science the remainder of al-Khwārizmī's work, notably a better study of positive rational numbers: one cannot be certain of the author of the translation, but it was compiled at the latest at the end of the twelfth century.[159]

If the first Latin evidence of medieval algebra is relatively well known and if its interpretation only poses minor problems from the point of view of the Arabic texts on which it is based, the situation is completely different as soon as one reaches the beginning of the thirteenth century some forty or fifty years after the translations mentioned. Two works dominated unequally during this period: the *De numeris datis* of Jordanus Nemorarius and the mathematical survey known as the *Liber abaci* of Leonardo Fibonacci (1202, revised in 1228). The problem of Arabic sources is posed here in a particularly testing manner: some elements which we put forward cannot pretend to clarify completely a question which deserves to be the subject of much more research.

We have stated earlier that the Arabo-Latin version of Euclid by Campanus of Novara partly inspired the *Arithmetic* of Jordanus Nemorarius and the *Liber de triangulis* of Pseudo-Nemorarius. In contrast the links between the books of Nemorarius and Fibonacci are less clear. One note for example that the problem

$$x + y = 10$$

$$\frac{x}{y} = 4$$

can be found in both the Latin versions II and III of al-Khwārizmī,[160] in the works of Abū Kāmil (end of folio 22v and beginning of folio 23r of the Arabic text) and in the *De numeris datis* (problem I, 19),[161] whilst Fibonacci expresses the same problem in the form[162]

$$x + y = 10$$

$$xy = \frac{x^2}{4}$$

Some examples suggest that Jordanus was inspired by Abū Kāmil, contrary to what the editor of *De numeris datis* has declared.[163] Thus the problem

$$x+y=10$$
$$x^2-y^2=80$$

can be found both in Jordanus's work (I, 24)[164] and in Abū Kāmil's work (folio 25 of the Arabic text), but not in the Latin translations of al-Khwārizmī, nor in the *Liber abaci* where one finds[165]

$$x+y=10$$
$$x^2-y^2=40$$

On the basis of just the problems II, 27–8 of Jordanus, which correspond to a Diophantine problem (*Arithmetica* I, 25), Wertheim has suggested an influence of al-Karajī.[166] We shall see that the same question is raised more seriously in relation to the *Liber abaci,* which moreover contains the same problems used by Jordanus;[167] it seems that one could refer again here to the work of Abū Kāmil. It is impossible to imagine that al-Karajī's important work (tenth and eleventh centuries), which in contrast with his predecessors provided a theory for algebraic calculation and the first explanation of polynomial algebra, was used only as a source for Diophantine examples.[168] The *De numeris datis* is a minor work on the ladder of the mathematical survey of Fibonacci. In the fourteenth century, however, Johann Scheubel thought it would be useful to revise it in the light of much more detailed works, in which perhaps the *Ars Magna* of Girolamo Cardano (1501–76) appeared describing, for the first time in the West, general solutions to cubic equations.[169] To determine more precisely what influence the work of Abū Kāmil had on those of Nemorarius and Fibonacci, one must wait until not only his *Algebra* is well known but also the Latin translation of his *Art of Calculation*[170] and his work in which Diophantine equations are explained in a much more detailed manner than previously.

In citing the *De numeris datis* and the *Liber abaci* we doubtless present a slightly distorted view of how the Latin West received the legacy of Arabic algebra before the thirteenth century, in so far as these Latin works already constituted a particularly auspicious result of the modest work undertaken by the twelfth-century translators. Little effort has been made so far to search in the Latin works for evidence of the earliest moments of the apprenticeship. We have already mentioned in connection with Indian reckoning that manuscript 15461 in Paris, copied on a Toledan model, allows us to date the *Liber Alchorismi* of John of Toledo at about 1143. The same manuscript contains an anonymous mathematical treatise whose only *incipit* conforms to numerous medieval works without hinting at Arabic sources.[171] This treatise can also be found in manuscript 7377A in Paris, a unique source even today of the Latin translation of Abū Kāmil's *Algebra*. An example taken from this treatise allows one to understand how the Arabic lesson was assimilated yet at the same time misunderstood during the early days of its discovery.

Wishing to demonstrate the rule for permutation of factors of a product, the author shows that for four numbers $a, b, g, d,$ multiplied two by two as $a.b=h, g.d=z, a.g=k$ and $b.d=t,$

$$h.z=k.t$$

He first states the following properties:

$$(a+d)b=h+t$$
$$(a+d)g=k+z$$

By using proposition VII, 18 of the Arabic Euclid, he obtains

$$\frac{t}{h}=\frac{d}{a} \qquad \text{and} \qquad \frac{z}{k}=\frac{d}{a}$$

and proposition VII; 19 of the Arabic Euclid enables him to demonstrate this proposition. Explicitly citing 'the third part of Abū Kāmil's *Algebra*',[172] the anonymous author proposes a second illustration using the property

$$\frac{h.z}{t}=k$$

and demonstrates his proposition in the same way. Proud of his completely new science and believing it possible to complete his source with 'an easier demonstration'[173] the author proposes

$$\frac{a}{b}=g \qquad \frac{d}{h}=z \qquad a.d=k \qquad b.h=t \qquad g.z=q$$

and at the price of a long pseudo-knowledgeable demonstration he shows that $k/t=q$.

The example, which is not at all exceptional, shows that, faced with the disruption aroused in these times by the contribution of Arabic works in the fields of Indian reckoning, Euclidean geometry and algebra, the medieval West had a difficult period of assimilation.

The Western works cited here were clearly left far behind by the *Liber abaci*. It is unnecessary to emphasize the central role occupied by Fibonacci in the scientific evolution in the West: since Cossali (1797), after a long period when he was forgotten, his role has been constantly recognized. Many works have made a study of Fibonacci's innumerable borrowings from Arabic sources.[174] Amongst these al-Khwārizmī, Abū Kāmil and al-Karajī figure regularly. From the author himself we know that he travelled to many places—North Africa, Egypt, Syria, Byzantia, Sicily and Italy.[175] One can imagine that his sources of information, besides the Latin texts which preceded him, were many and varied. But the answer to the question whether this information came from the original Arabic texts or from Latin translations must remain unknown. Fibonacci made use of a Latin translation of al-Khwārizmī's *Algebra* and the vocabulary he used shows that this was Gerard of Cremona's translation: thus the Latin words *regula* and *consideratio* translating the Arabic term *qiyās* ('reasoning') are used by these two authors in the same circumstances.[176] Also, one does not find any evidence in *Liber abaci* of al-Khwārizmī's *Algebra* which is not perceptible in Gerard of Cremona's very faithful translation. Occasional studies have likewise shown the influence of Abū Kāmil's *Algebra* in Fibonacci's work. Thus Levey stressed the identical nature of the thirty-nine problems in the two works,[177] but no exhaustive study exists on this question. For example, another series of problems in which Fibonacci translated the double meaning of the Arabic word *māl* by *auere* ('property, fortune') and *census* ('square') shows beyond question the influence of Abū Kāmil.[178] It is clear also from the problem[179]

$$2\sqrt{x}+\sqrt{\left(\frac{x}{2}\right)}+\sqrt{\left(\frac{x}{3}\right)}=x$$

If one compares the text of *Liber abaci* and the Arabic and Latin texts of Abū Kāmil's work, one can clearly see together the source and the manner in which it has been used.

Abū Kāmil (*Algebra,* Arabic text f.47v and Latin text f.88v):

> And if we say that two roots of a thing and the root of its half and the root of its third are equal to a thing, how much is this thing? Suppose a square to be this thing, *and* say that two things and the root of one-half of the square and the root of one-third of the square are equal to a square. Thus a thing is equal to two and the root of one-half and the root of one-third. And this is the root of the thing, and the thing is four and one-half and one-third, and the root of eight, and the root of five and one-third, and the root of two-thirds.

Fibonacci (new critical edition of *Liber abaci*):

> There is a certain thing to which two of its roots and the root of its half and the root of its third are equal. Suppose instead of the thing a square. And since two things and the root of one-half of the square and the root of one-third of the

square are equal to a square, make the aforementioned square *ac* which is a square (Figure 16.5), and two roots of this square the area *dg,* and the root of one-half of the square the area *eh,* and the root of one-third of the square the area *bf.* Thus *cg* will be two, *eg* will be the root of one-half of a drachm, and *be* will

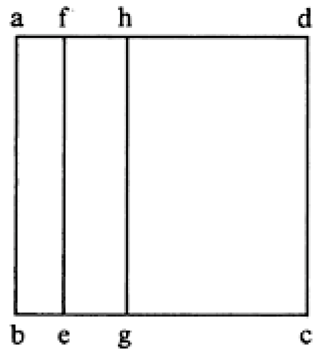


Figure 16.5

be the root of one-third of a drachm. Thus the whole *bc,* which is a thing, will be two and the root of one-half and the root of one-third. Multiply this by itself and it will result four and five-sixths, and the root of eight, and the root of five and one-third, and the root of two-thirds of a drachm, for the quantity of the square, i.e. the desired thing.

Although he does not mention it Fibonacci used his knowledge of the Arabic Euclid to solve the problem posed above (*Elements* II, 1). This is what distinguishes him from Abū Kāmil (whose influence here as elsewhere, however, appears to be undeniable) for whom the demonstrations by areas are only ever supplementary illustrations. This method of solution in the *Liber abaci,* although not systematically applied, makes Fibonacci's algebra slightly inconsistent, the influence of which is obvious in the *Quadripartitum numerorum* of John of Murs (first half of the fourteenth century), largely used by Regiomontanus.[180] In our state of knowledge, it is often difficult to tell what comes from Fibonacci's work and what is from Arabic sources. Thus the solution of numerical equations supposes a mastery of algorithms allowing one to extract numerical roots. Before showing by numerous examples how to extract a cubic root in a manner corresponding to the formula

$$\sqrt[3]{(a^3 + r)} \sim a + \frac{r}{3a(a+1)+1}$$

Fibonacci claimed this as his discovery.[181] It was nothing more, however, than the 'conventional approximation' according to the expression of al-Ṭūsī (second half of the twelfth century), known at least since the time of Abū Manṣūr (d. 1037) and distinct from the approximation

$$\sqrt[3]{(a^3+r)} \sim a + \frac{r}{3a^2+1}$$

The latter is that of, for example, Kushyār ibn Labbān (*c.* 1000) and his pupil al-Nasawī (eleventh century).[183] Did Fibonacci in fact rediscover an approximation used well before his time, or did he merely take it from Arabic sources, none of which are explicitly quoted in his book? One cannot answer this question factually. We note, however, that occasional studies have shown the similarity between Fibonacci's propositions and those of the Arabic authors who preceded him: for example the problem of linear congruences where Fibonacci's solutions merely summarize those of an *Opuscule* by Ibn al-Haytham.[183] But the traditional affirmation since Woepcke[184] that Fibonacci largely used the *Fakhrī* of al-Karajī deserves reexamination in the light of the *Algebra* of Abū Kāmil, for those issues which concern the *Liber abaci*. We remember that the analysis of Fibonacci's other works which contain algebraic problems[185] noted some similarities with the works of al-Karajī and al-Khayyām.[186]

We cannot detail here a history of algebraic equations in the medieval West between the first discoveries due to the *Algebra* of al-Khwārizmī and the general solutions of quadratic, cubic and biquadratic equations which appeared in the *Ars Magna* (1545) of Girolamo Cardano. The thirteenth-and fourteenth-century works which could contain equations revealing terms of greater power than two are not yet well known however. Equations of the type

$$ax^{n+2p}+bx^{n+p}=cx^n$$

are known in fifteenth-century works such as the *Triparty* of Nicholas Chuquet (1484)[187] or the *Summa* of Luca Pacioli (1494),[188] and especially in the numerous works of the sixteenth century.[189] But the use made by the medieval West of the lesson begun in the twelfth century and the opportunity seized from this to realize a link with a legacy often stretching to Bede the Venerable or to Alcuin are never as discernible as in the manner in which all the authors treated the problems of everyday life or the problems of recreational mathematics.[190] Thus we have indicated previously that the linear equation with two unknowns

$$x+y=10$$

$$\frac{x}{y}=4$$

corresponding to problem I, 2 of the *Arithmetica* of Diophantus of Alexandria, appears in the works of al-Khwārizmī and Abū Kāmil and later those of Leonardo Fibonacci[191] and Jordanus Nemorarius. Fibonacci explained a variation on the same problem where

$$\frac{x}{y}=\frac{2}{3}$$

which is in fact identical to the problem described by al-Karajī.[192] Without wishing to make here an exhaustive analysis of the problem in medieval works, we note only that it occurs in several variations in the following works.

Fourteenth century

An anonymous *Libro d'abaco;*[193] the *Trattato d'aritmetica* by Paolo Dagomari;[194] an anonymous Italian treatise on commercial arithmetic[195]

Fifteenth century

Algorismus Ratisbonensis[196] and its revision;[197] the *Trattato d'abaco* by Piero della Francesca;[198] the *Tractate d'abbacho* by Pier Maria Calandri;[199] an anonymous arithmetical treatise (*c.* 1480);[200] the *Triparty* by Nicholas Chuquet;[201] the German commercial arithmetic of Johannes Widmann (1489);[202] the Italian arithmetic of Francesco Pellos (1492)[203]

Sixteenth century

The *Summa* by Francesco Ghaligai (1521);[204] the *Coss* by Christoff Rudolff (1525);[205] the *Coss* by Adam Riese;[206] the *Practica arithmeticae* by Girolamo Cardano (1539);[207] the arithmetic of Niccolò Tartaglia (1556)[208]

The medieval authors had barely assimilated that the series of surprising developments by al-Khwārizmī's successors were the beginnings of algebra. This was only recognized as an autonomous science much later in the West and remained an integral part of the solution to problems of commercial arithmetic during the Middle Ages, especially in Italy and Germany where its use was more developed. Fibonacci escapes this summary judgement; nevertheless his work only contains an occasional indication of the influence of al-Karajī, al-Khayyām or Ibn al-Haytham. The new bases of algebra were laid down with François Viete (1540–1603), carrying Western science into the modern era.

NOTES

1 *Codex Vigilianus* of the Escurial, written at the monastery of Albelda, in the valley of the Ebre. Cf. Smith and Karpinski (1911:137–9).
2 On this subject see Sezgin, III, pp. 266, 295–7, 304–7; the synthesis of Schipperges, pp. 17–54; and several articles by Creutz.
3 However, we do not believe that Indo-Arabic numerals spread to the West via abacus

counters, but rather through the manuscripts on Indian reckoning. On this subject see Allard (1987) and Beaujouan (1948:301–3). Note that on two occasions Gerbert mentions a pamphlet *De multiplicatione et diuisione* by Joseph the Wise (Josephus Hispanus) which is now lost. This work was doubtlessly limited to describing the two most difficult operations one could achieve with the aid of an abacus.

4 For the true name of the Arabic author and the contents of his work on algebra, see Rashed (1984:17–29). al-Khwārizmī wrote at least two works, both lost: one specifically devoted to Indian reckoning (*Ḥisāb al-Hind*) and the other, cited by Abū Kāmil, which dealt with arithmetical problems *Kitāb al-jamʿ wa al-tafrīq*).

5 All these versions are edited and translated in Allard (1992:1–224.

6 A detailed history of these texts appears in Allard (1992: xxxv–lxv).

7 On this subject see Burnett (1991:1–37).

8 A detailed study of these texts appears in Allard (1991; 1992: i–xxxv, 225–70).

9 Boncompagni (1857a); Vogel (1963); Youschkevitch (1964c); Allard (1992). The beginning of this text has already appeared in Halliwell (1841:73 n3). Note, however, that the Cambridge manuscript (University Library Ii.6.5) traditionally dated as thirteenth century or even fourteenth century, seems most probably to have been copied around 1150 according to recent studies by Thomson still in progress.

10 These parts have been edited in a provisional manner by Dickey.

11 This identity is still the subject of discussion. Lemay (1987:418) firmly contends that John of Seville, John David and Avendauth are one and the same person, contrary to the opinion of d'Alverny (1954:19–43). Lemay (1977:456) even maintains that the author of the first version of *Liber Ysagogarum* was certainly Petrus Alfonsus, but this attribution must be abandoned.

12 A detailed study of these numerals appears in Allard (1987).

13 This thesis on the Visigothic origins of numerals is developed by Lemay (1977:435–62).

14 Text cited by d'Alverny (1982:440), after Vernet (1979:568).

15 Cf. Allard (1992:151–5, 160–3).

16 *Ibid.,* pp. 163–6.

17 A supplementary proof, if necessary, that *Liber Pulueris* does not stem from *Liber Alchorismi,* but that they share a common source.

18 *Similiter etiam idem est superioribus quod de divisione docet dicens* ('That which he teaches concerning division is similar to that seen above, as he said'). Cf. Allard (1992:168).

19 Such as Sanad ibn ʿAlī, al-Ṣaydanānī, Sinān ibn Fatḥ, al-Karābīsī, al-Anṭakī, al-Kalwadhānī. One could add here other authors whose works are known today. Cf. Levey and Petruck (1966); al-Uqlīdisī (English translation) (a list of works which are actually known appears on pages 3–5).

20 However, one can make no comment about *Dixit Algorizmi* concerning this matter which is missing from the Cambridge manuscript.

21 See Allard (1992:9–10). One successively displaces the multiplier one row to the right; the underlined numbers are meant to be erased and replaced by those which are above them. On the same subject, in versions I and II of the *Liber Ysagogarum Alchoarismi* 1024 is multiplied by 306, in version III of the *Liber Ysagogarum*

Alchoarismi 406 is multiplied by 204, and in *Liber Alchorismi* as in *Liber pulueris* 104 is multiplied by 206.

22 The invention of these is attributed to the Indians by *Dixit Algorizmi* and *Liber Alchorismi,* whilst *Liber pulueris* attributes it to the Egyptians; *Liber Ysagogarum* has no comment on this question.

23 Cf. Allard (1992:146–8). The fractions expressed in this system are connected by the word *'et'* (and). Only the *Liber Alchorismi* contains examples of normal successive fractions.

24 Cf. Allard (1992:158–9).

25 See for example Al-Uqlīdisī (English translation), pp. 60–3.

26 Cf. Allard (1992:59–61, 206–24).

27 *Aut si hoc facere uolueris, denominabis illud quod remanserit scilicet quota pars sit illius numeri per quem dividis* ('Or if you wish, you may give the remainder a denominator, of which the value will be determined by the number with which you divide').

28 We cite here the general formula of **al-Samaw'al,** which is identical to that of the Latin texts, as Rashed (1984:121) notes.

29 Rashed (1984:93–145).

30 In principle this method conforms to that of Benedict (1984); but the numerous errors contained in this work make its use dangerous. Cf. Allard (1978).

31 One would like to consider that this list is not exhaustive: several arithmetical texts where the first traces of the influence of the algebra of al-Khwārizmī or Abū Kāmil are revealed are contained in Latin manuscripts which are still unedited.

32 Cantor (1865).

33 Karpinski (1921).

34 Waters (1928).

35 Halliwell (1841:1–26); Curtze (1897:1–19).

36 Halliwell (1841:73–83).

37 Allard (1978:128–40).

38 Compare with the complementary notes to the edition of *Dixit Algorizmi, Liber Ysagogarum, Liber Alchorismi* and *Liber pulueris* in Allard (1992:225–48).

39 Levey and Petruck (1996).

40 Al-Uqlīdisī (English translation).

41 Halliwell (1841:1–26); Curtze (1897:1–19). The two hundred odd manuscripts known today and the numerous successive editions between 1488 and 1568, listed by Smith (1908:31–3) adequately demonstrate the popular success of this work.

42 Halliwell (1841:73–83). There are a very large number of manuscripts of this work and several translations into the vernacular, of which the most ancient, in French, seems to date from the thirteenth century.

43 According to author, these are used *in Tabula dealbata in qua littere leuiter deleantur* ('on a whitened tablet where the written characters can easily be erased'). Cf. Boncompagni (1857c: I, p. 7).

44 Thus, adjusted as above to the usage in this chapter, the process of multiplication by erasement is given the name 'multiplication in the form of a galley' by Bienewitz.

45 Boncompagni (1857b:119–20).

46 Curtze (1897:9).

47 Al-Uqlīdisī (English translation).

48 Boncompagni (1857c:I, p. 12). We do not aim at all to show that Fibonacci used such and such an Arabic text, for example the arithmetic of Al-Uqlīdisī, but only that some calculation procedures used for a long time in the Arab world were picked up by the medieval West. This could have been by a knowledge of some texts as well as by contacts with the Muslim world.

49 Allard (1977:83–7).

50 Allard (1981:56–74).

51 Cf. al-Uqlīdisī (English translation), pp. 136–7. The diagram which we propose is the explanation of one of al-Uqlīdisī's methods for adding 'houses'; the diagonals do not appear in the diagrams in the text itself.

52 Boncompagni (1857c: I, p. 19). In Florence the same method was called 'apricot cake' *(per bericuocolo)*.

53 One even meets the *gelosia* method in a Byzantine manuscript written without doubt at the end of the fourteenth century. Cf. Allard (1973:120–31).

54 This is the 'fisherman's net' *(shabaka)* method of Arabic authors.

55 *Quemadmodum potest super datam directam terminatam lineam trigonum aequilaterum constitui* ('To construct an equilateral triangle on a given straight line'). Cf. Willis (1983:258).

56 Known from the example of Ralph of Liege (*c.* 1025) under the name *Podismus,* a possible reference to the work of Marcus Junius Nipsus.

57 It is also established that this author's value of π is given as $(9/5)^2=3.24$. His pamphlet *De quadratura circuit* dedicated to Herman, archbishop of Cologne (1036–56) (Folkerts and Smeur 1976), did not result from a work of geometry but from commentary of Boethius on the *Categories* of Aristotle; Archimedes' approximation is here considered to be an exact value, and the formula for the area of a circle handed down from the *Agrimensores,* i.e. eleven times the square of the diameter divided by 14, corresponding to a value for π of approximately 3.1429, is also accepted as a fact.

58 Folkerts (1970:69).

59 According to the traditional term after Bubnov, or *Demonstratio artis geometricae* in Blume *et al*. (1848:377–412), and justifiably qualified by Tannery (1900) as 'pseudo-geometry'.

60 On the contents of the work, see Folkerts (1970:69–104).

61 See for example Halleux's synthesis (1977:489–96) on the Liege mathematicians of the tenth and eleventh centuries. Or otherwise a fragment of a sixth-century Latin translation of the *Elements* of Euclid in a palimpsest from Verona is evidence of a much better knowledge of Euclidean geometry, but there certainly existed very few items on this subject between the ninth and twelfth centuries in the compilations where the extracts from *Agrimensores* were dominant. Cf. Geymonat (1964).

62 Weissenborn (1880).

63 Björnbo (1905).

64 Clagett (1953).

65 Murdoch (1968).

66 These works, founded on the study of numerous manuscripts, are due principally to Lorch, Burnett, Folkerts and Busard.

67 The most well-known Arabic version of Euclid is that of al-Ṭūsī, later than the Latin works studied here. There is also a version falsely attributed to al-Ṭūsī and printed in Rome in 1594.
68 Cf. De Young (1984) and Kunitzsch (1985).
69 These versions are classed as I, II and III after the fundamental article by Clagett (1953). On this question see Busard and Folkerts (1992).
70 Folkerts (1987:63). We gave details in the first section of the contents of the *Liber Ysagogarum Alchoarismi* and of those parts which could have come from Adelard's work. It appears that the thesis advancing Adelard of Bath as the author of the *Liber Ysagogarum Alchoarismi* cannot be upheld.
71 Busard (1968a). The attribution to Herman of Carinthia is traditional after the work of Birkenmajer on the library of Richard of Fournival. Cf. Haskins (1924:50).
72 Busard (1984). Other manuscripts have been cited by the editor since this edition. See Busard (1985:130–1).
73 See Kunitzsch (1985) and the demonstration of Lorch (1987:47–53).
74 The recent edition of Busard and Folkerts (1992) gives the most complete situation concerning the first Latin translation of Euclid. We have only mentioned some well-known elements here.
75 *Pinguis Minerua* in XI, 21 (*De Amicitia* V, 19). The same quotation appears in the *De eodem et diuerso* of Adelard of Bath.
76 Proposition X, 42 and the introduction to book X.
77 We only put forward this hypothesis with the greatest caution: to quote word for word from the manuscripts, *'ex* (or *in*) *Ocrea Johannis'* which makes the simple reference to a John or a Nicholas Ocreatus syntactically difficult. An answer to this problem can be found in the first three folios of the twelfth-century manuscript, Trinity College, Cambridge, R 15 16, where 'Alardus' and ' Johannes' are cited as geometrists. Other references at the end of book X to 'Lincol' (Lincol‹niensis›?), 'Zeob', 'Rog' (Rog‹erius›?), 'Hel' (He1‹iensis›?) remain a mystery.
78 According to Folkerts (1987:64 n55).
79 Besides characteristic expressions such as XIII, 7: *'wa delicah me aradene en nubeienne',* one often finds terms such as *hypothenusa, ysosceles* etc. used, which never appeared in version I.
80 But Bacon certainly used manuscript 16648 of the National Library of Paris, whose colophon talks of an *'editio'.* The preface to the text is edited by Clagett (1954:273–7). A disparate collection of geometrical problems also exists under the title *Bathon (Bachon?) Alardus in 10 Euclidis* in the manuscript Conv. soppr. J IX 26 (folios 46–55) of the National Library of Florence. The editor attributes this to Roger Bacon. Cf. Busard (1974).
81 Compare the detailed study on this question and Folkerts's conclusions (1987:66–8).
82 Cf. Busard (1984:xi–xii; 1985:133–4).
83 Busard (1984).
84 Thus propositions I, 45; VIII, 24, 25; X, 21, 22; and the aphorisms of VIII, 14, 15. Alī these elements are omitted in the versions of Adelard of Bath and Herman of Carinthia.
85 Curtze (1899:1–252).

86 *Ibid.,* pp. 252–386.
87 Junge (1934:1–17).
88 Cf. Murdoch (1966). The first complete Latin translation of the Greek text appeared in Venice in 1505; however, it is the edition of Federico Commandino (Pesaro, 1572) which was used as the basis of all subsequent editions until the beginning of the nineteenth century.
89 Cf. Victor (1979).
90 Cf. Tummers (1984) and Hofmann (1960).
91 This example is quoted by Busard (1985:139–40, 153–4).
92 *Propositis tribus quantitatibus eiusdem generis proportio prime ad tertiam producitur ex proportione prime ad secundam et proportione secunde ad tertiam.* Cf. Busard (1985:153 n47).
93 Clagett (1964:II, pp. 16–18).
94 *Proportio ex proportionibus constare dicitur quando proportionum quantitates in se ipsas multiplicate fecerint aliquam.*
95 *Dicitur quod proportio ex proportionibus aggregatur quando ex multiplication quantitatis proportionum, cum multiplicantur in seipsas, prouenit proportio aliqua.*
96 Schrader (1961:125).
97 Boncompagni (1857c: I, p. 119).
98 Crosby (1955;74).
99 Busard (1971:193–227).
100 Curtze (1887:45–6 n29).
101 Busard (1971:215 n30).
102 Clagett (1964:II, pp. 13–15). The study, dedicated to the great translator of the thirteenth century, William of Moerbeke, is mostly inspired by the *Optics* of Ibn al-Haytham (Alhazen) and constitutes an important link in the diffusion of Greco-Arabic optics; Kepler makes a reference to it, in the title itself of the edition of his *Optics* of 1604.
103 North (1976:I, p. 60).
104 McCue (1961:25–6 n46).
105 Crosby (1955:76).
106 Busard (1985:140).
107 Curtze (1899:1–252). The influence of al-Nayrīzī in the work of Roger Bacon is not limited to his commentary on Euclid: one can also find it in the unedited part of the *Communia Mathematica* contained in the Digby 76 manuscript at Oxford. However, Albert the Great's sources are not all formally identified: one often finds in the text such statements as *translatio ex greco, translatio ex arabico* which indicate that a Latin translation of a Greek text was known to the author and that he distinguished it from Arabic sources.
108 In the medieval sense of the term this meant the addition, to a statement and its demonstration, of other demonstrations, of corollaries or of additional theorems. Later on we shall see another example relating to the trisection of an angle.
109 To the statement and the demonstration I, 1 of the *Elements* Campanus adds two demonstrations corresponding to those of al-Nayrīzī. Cf. Clagett (1953:29 n31(4)). See also Murdoch (1968:80 n41, 82 n53, 89 n84, 92 n100).
110 For example in *Elements* V, 16.
111 Thus, I, 48 of Campanus (folio 10 in the 1482 edition) agrees with IV, 17 of *De*

triangulis in the edition of Curtze (1887:37). In V, 16 Campanus quotes the last of the definitions included in the second book of *Arithmetic* (in the old edition of Jacques Lefèvre d'Etaples). Busard, taking up the project of Grant, completed an edition of the *Arithmetic* of Jordanus, which so far was seriously lacking. We just indicate here that there are many clear differences between the vocabulary of the two authors. For example in *Seriem numerorum in infinitum posse extendi* and *Nullum numerum in infinitum decrescere* (*Petitiones* 3 and 4 of book I) the verbs *extendi* and *decrescere* of Jordanus are replaced by *diminui* and *procedere* respectively in Campanus.

112 Thus, to take a single example, Philippe Elephant, a doctor from Toulouse in the middle of the fourteenth century, wrote a *Mathematica* in which the geometrical part is largely inspired by Campanus. Cf. Cattin (1969).

113 Busard (1965). To define, for example, the nature of a cylinder *(columna rotunda)* or of a cone *(piramis rotunda)* before finding their areas, the author explicitly cites definitions 11 and 9 from book XI of Campanus (definitions 21 and 18 in the Greek text).

114 Compare the following paragraph on algebra. So far we have no trace of many of the numerous Latin translations of Arabic works completed during the twelfth century, and Fibonacci's work does not show his knowledge of Arabic. It is in this context that one must understand the conclusions of the better authors, such as that of Rashed (1984:259): ' No-one is unaware that Fibonacci was in direct contact with the Arabic mathematics'.

115 Certain parts of the *Practica geometric* also show perhaps the influence of the *Metrics* by Hero of Alexandria. Boncompagni (1857c:II) and Arrighi (1966).

116 Again this is a 'commentary' in the medieval sense of the word.

117 The text under consideration is the first of the two texts published by Woepcke (1891).

118 According to the conclusions of Archibald (1915:11).

119 Clagett (1964:I–V). The seventh chapter of volume I (pp. 558–63) summarizes the conclusions of the author on the Arabo-Latin tradition of Archimedes. These conclusions are completed in volumes III to V.

120 Clagett (1956).

121 Clagett (1964:I, pp. 439–557). However, one notes that the editor shows many instances of the influence of the Greek text in this work.

122 Which, from an algebraic point of view, comes back again to the solution of a problem of the third degree.

123 *Kitāb al-mākhūdhāt,* translated by Thābit ibn Qurra and commented on by al-Nasāwī; the translation and the commentary form the basis of an edition by al-Ṭūsī.

124 Cf. Heath (1921:I, pp. 235–44).

125 According to the term that Roger Bacon gives in his *Communia Mathematica*.

126 Clagett (1964:I, pp. 223–367).

127 We omit the demonstration which appears in the text.

128 Compare the detailed explanation given by Clagett (1964:I, pp. 666–8).

129 Clagett (1964:I, pp. 672–7). On the author of *De triangulis,* see the conclusions of Clagett (1964:IV, pp. 25–9; V, pp. 323–4).

130…*mihi nequaquam sufficit dicta demonstratio, eo quod nihil in ea certum reperio* (‘the demonstration given does not satisfy me, since I do not find any certainty there’).

131 Clagett (1964:IV, pp. 19–20, 25–6, 28–9).

132 Clagett (1964:I, pp. 678–81).

133 Area=$[p(p-a)(p-b)(p-c)]^{1/2}$, where p represents the semi-perimeter and a, b, c are the sides. The formula is attributed by al-Bīrūnī to Archimedes and is certainly earlier than Heron.

134 Clagett (1961:IV).

135 One can find a very good explanation of the transmission of the *Elements* of Euclid in Murdoch (*Dictionary of Scientific Biography,* 1971:IV, 437–59). This account is completed with the recent studies cited in this chapter.

136 The fundamental operations of calculus are still taught today according to the methods which come from the Italian commercial arithmetic of the fifteenth century, which stem largely from the methods of Indian reckoning revealed by Arabic works. Until very recently, part of Euclidean geometry formed an important element of secondary school education in most European countries: we have shown its medieval origins.

137 The repeated calls of pioneers such as Paul Tannery or Georges Sarton have not always been heard.

138 See the excellent synthesis of Beaujouan (1966:598). On Western knowledge of the Greek text of Diophantus see Allard (1982).

139 *Kitāb al-muktaṣar fī ḥisāb al-jabr wa al-muqābala*. On the real meaning of this work see Rashed (1984:17–29).

140 Boncompagni (1857b:II, pp. 93–136). Several manuscripts that we have used for chapters on Indian reckoning do not contain this part, on which an edition is in preparation.

141 Re-examined by Juschkewitsch (1864b:342). The authenticity of the magic square is doubted. The studies which we have undertaken on this part of the text do not allow a reconstruction of the tale at this time.

142 Thus the definition *unitas est origo et pars numeri,* distinct from those of the Latin translations of Euclid.

143 And not *glaba mutabilia* as the terrible wording of the Parisian manuscript states, as transcribed by the editor. One can search in vain for any meaning in the following edited text: *queres* (‘you will search’) for *que res* (‘that square’), *tocius* (‘of the sum’) for *tociens* (‘a number of times’) etc. We cite in passing some extracts re-established with the help of manuscripts of *Liber Alchorismi* and we refer to the text itself as version I.

144 *Aut que res cum tociens radice sua efficiat numerum* (i.e. $x^2+px=q$); *aut que res cum tali numero efficiat tociens radicem* (i.e. $x^2+q=px$); *aut que tociens radix cum tali numero efficiat rem* (i.e. $x^2=px+q$).

145 Karpinski (1915) and Hugues (1989); this text is cited here as version II.

146 See for example Karpinski (1915:89–9 and n2).

147 Björnbo (1905:239–41).

148 Libri (1838:I, pp. 412–35) and Hughes (1986); this text is cited here as version III.

149 Boncompagni (1951:412–35).

150 One notes several differences in the vocabulary even of the translations: the square *(māl)* is rendered as *res* (text I), *substantia* (text II) and *census* (text III and revision); the root of a square *(jidhr)* is rendered as *radix* (texts I, II and III), as *radix* or *res* (revision of text III); the plurality of units *(dirham)* is rendered as *numerus* (text I), *drachmae* (texts II and III), *dragmae* or *unitates* (revision of text III). One would expect the word *res* in text I to translate the word *shay'* of al-Khwārizmī to express an unknown quantity and the word *numerus,* which certain Latin algebraists use later as the Diophantine word *res* ('thing') to designate an unknown quantity, is a less faithful translation than the word *dragmae*.

151 The translation of the text of the modern edition by Musharrafa and Ahmad is taken from Rashed (1984:23): we have only the old edition by Rosen (1831). The text proposed for version II is that of the edition in preparation. In contrast to Karpinski (1915:76), we contend that the parts in square brackets are the interventions of Scheubel, whose sources were numerous. Cf. Hugues (1972:224–5). We note in passing the quality of Gerard of Cremona's translation (version III).

Version II: *Una radix substantiae simul etiam medietas radicum [quae cum substantia sunt] pronunciatur, adiectione simul et diminutione abiectis* ('One makes known a single root of the square which is at the same time half of the roots [which accompany the square], at the same time rejecting both the excess and the shortfall').

Version III: *Tum radix census est equalis medietati radicum absque augmento et diminutione* (Thus the root of the square is equal to half of the roots, without any excess or shortfall').

Revised version III: *Erit radix census equa dimidiis radicibus* ('The root of the square will be equal to the roots divided by two'). For the example by Ibn Turk, see Sezgin (1974:V, p. 242).

152 *Habebitur pro radice census numerus medietatis radicum* ('For the root of the square one will have the number of half of the roots'). Cf. Boncompagni (1857c:I, p. 406).

153 Hugues (1981:100–1). For example Jordanus applies the second trinomial equation of al-Khwārizmī by solving $x^2+8=6x$.

154 Corresponding in one case to al-Khwārizmī's demonstration and in others to those of Abū Kāmil.

155 Cf. Boncompagni (1857c:I, pp. 406–9).

156 Cf. Arrighi (1964:85–91).

157 Cf. Busard (1868b).

158 Busard (1868b:86–124).

159 Karpinski (1911). On the content of the book by Abū Kāmil, see Juschkewitsch (1976:52 *et seq*.) and Levey (1966). The hypothesis of Sarton (1927–31:II, p. 341) attributing the translation to Gerard of Cremona has not been demonstrated so far. I am preparing a critical edition with Rashed of the Arabic and Latin texts of Abū Kāmil.

160 Karpinski (1915:105–6) and Libri (1838:I, p. 276).

161 Hugues (1981:64).

162 Boncompagni (1857c: I, p. 410).

163 Hugues (1981:12).
164 *Ibid.,* p. 62.
165 Karpinski (1915:111), Libri (1838:I, p. 279) and Boncompagni (1857c:I, p. 411).
166 Wertheim (1900:417).
167 Boncompagni (1857c:I, p. 410).
168 On the work of al-Karajī, see Rashed (1984:31–41).
169 Cf. Hugues (1972:224–5).
170 We have undertaken an edition of the anonymous Latin translation of *Kitāb al-ṭarā'if fī al-ḥisāb* ('The book of curious items in the art of reckoning').
171 *Omnium que sunt alia sunt ex artificio hominis, alia non…*('Of all things which exist, some result from man's ingenuity, others do not…').
172 *Hoc etiam monstrabitur ex eo quod dixit Auoquamel in tercia parte libri gebleamuqabala* ('This will be shown equally well by what Abū Kāmil says in the third part of his book of *al-jabr* and *al-muqābala*'). It seems that this is the first explicit citation in the West of the work of Abū Kāmil.
173 *Inducam probationem de eo quod dixit Auoquamel multo faciliorem ea quam ipse posuit* ('I will introduce proof of what Abū Kāmil said, much easier than that which he put forward').
174 On this point see the bibliography of Vogel (1971:613).
175 Boncompagni (1857c:I, p. 1).
176 On this point see the demonstration of Miura (1981:60).
177 Levey (1966:217–20).
178 Boncompagni (1857c:I, pp. 442–5).
179 *Est quoddam auere cui due radices et radix medietatis eius et radix tercie eius sunt equales. Pone pro ipso auere censum…* The text transcribed by Boncompagni (1857c:443) is quite incorrect and does not allow one to understand the problem. We are preparing a critical edition of *Liber abaci*.
180 L'Huillier (1980:201–6; 1990:56–9).
181 *Inueni hoc modum reperiendi radices secundum quod inferius explicabo* ('I have discovered this way of finding roots which I shall explain later'). Cf. Boncompagni (1857c:I, p. 378).
182 Contrary to the affirmation of Juschkewitsch (1964b:246). Cf. Rashed (1984:153–4 n3) and Sharaf al-Dīn al-Ṭūsī, pp. LXXX–LXXXIV.
183 Rashed (1984:234 n12).
184 Woepcke (1853:29).
185 Boncompagni (1857c:II, pp. 227–9).
186 Notably for the resolution of a cubic equation of al-Khayyām ($x^3+2x^2+10x=20$) in the *Flos*. See also the considerations of Rashed (1984:279) concerning the lemma said to be of Fibonacci (the condition of a natural prime integer), well attested in earlier Arabic works.
187 Marre (1880/1:807–14). The last equation is $2x^{10}+243=487x^5$ (p. 814).
188 Pacioli (1494), folio 149r°.
189 Cf. Tropfke (1980:442). In the same work one can read (pp. 443–4) an interesting analysis of a manuscript of Regiomontanus.
190 See the systematic analysis in Tropfke (1980:513–660).

191 Although with this particular expression $xy=x^2/4$.

192 Woepcke (1853:92) and Boncompagni (1857c:I, p. 410). Repeated examples of this type could become a proof, independent of Diophantine equations, that Fibonacci had knowledge of the book of al-Karajī.

193 Arrighi (1973:112).

194 Arrighi (1964:58).

195 Vogel (1977:24).

196 Vogel (1954:72).

197 Curtze (1895:52).

198 Arrighi (1970:92).

199 Arrighi (1974:89).

200 Wappler (1886–7:16).

201 Marre (1880/1:635).

202 Folio 37^r.

203 Folio 49^v.

204 Folio 57^v.

205 Folio 8^v.

206 Berlet (1892:41).

207 Ch. 66, problem 62.

208 Folio 266^r.

17
Musical science

JEAN-CLAUDE CHABRIER

APPROACHES OF ARABIC MUSICAL SCIENCE

Since the birth of Islam, Arab scholars have had the idea of comparing the native empirical musical practices with the musical theories to which they have access, i.e. more precisely to the theoretical systems in use among the Greeks, the Byzantines, the Lakhmids of the Kingdom of Hīrā, the Sasanids of Iran, and more essentially to the Greek theories. But if the observation led them from musical practices to theoretical systems, the works that they compiled have proceeded rather in the opposite sense, from theory to practice.

In fact, in general one finds the following in these works.

Basic theoretical scale of the available sounds

In the first place, an elaboration test, most often on the fingerboard of a short-necked lute, the *'ūd*, sometimes on the fingerboard of a long-necked lute, the *ṭunbūr*, exceptionally on a hurdy-gurdy, the *rabāba,* of a theoretical scale of sounds (in German, for example, 'ton system') defining the location of all the accessible virtual sounds from the low-pitched to the high-pitched and the value of the intervals that separate these sounds. We note in passing that the modal systems of Greek antiquity were produced, in general, on the lyre, and that the modal systems of Islam were produced most frequently on the lute.

It is necessary to understand clearly that the theoretical scale of sounds is a series or a sort of framework of consecutive accessible sounds situated from the low-pitched to the high-pitched inside one or several octaves from which the musician scholars could select the intervals or calibrate the fingering degrees constituting the genres and the modes. In general, the theoretical scale of sounds is made up of twenty-four fingering degrees per octave in order to constitute a heptatonic musical style.

The thing to do is therefore to adopt, adapt or calculate a theoretical acoustic system and a temperament in accordance with the manner of playing.

Trichord, tetrachord, pentachord genres

Once this acoustic system is adopted, adapted or calculated, the theoretical treatise passes to the study of tetrachord genres, most frequently on the fingerboard of the *'ūd*, defining in this way the places to hold for the fingers of the left hand that determine the pitch of the sounds by the selection and shortening of the strings. One thus defines the places for the index finger, the middle finger, the ring finger and the little finger. At this second stage, it is not necessary to situate all the available virtual sounds, but to select those sounds which are going to constitute the genre (*jins,* pl. *ajnās*), for example the major genre with major third or the minor genre with minor third, or again, between the two, the neutral genre with neutral third. The definition for fingering degrees is essential, because, according to whether one is playing a minor third or a major third, one uses the middle finger or the third finger.

Musical modes (*maqām,* pl. *maqāmāt*)

Then the treatise passes in general to the nomenclature of the different musical modes that can be described in the music envisaged and explains the way in which these modes can be realized on the frets of the reference instrument. The vast majority of the Arabic, Iranian, Turkish and comparable musical modes are made up of modes called heptatonic, i.e. comprising seven fingering degrees per octave, like the 'Western' modes. The differences found between these two types are in the nature of the intervals separating the fingering degrees.

This elaboration of the acoustic systems destined to reconcile the native practices of music and the theories derived from ancient Greece and then from Europe has exercised the imagination of great numbers of scholars and thinkers from the ninth century to the present. From this preoccupation a large number of treatises have appeared essentially putting forward systems of acoustics. It is interesting to discover that the ensemble of these treatises (of which a good many have been translated into Western languages) can serve as a grounding to a history of Arabic musical science, because, with a careful reading, one discovers that the elaboration of acoustic systems, even though dominated clearly by the Pythagorean system, evolved from the ninth to the twentieth century in a fascinating fashion.

We retain therefore as an essential criterion of the approach of Arabic musical science (or Arabo-Islamic in a broad sense) the comparative study of the evolution of acoustic systems successively proposed since the ninth century, because this criterion applies essentially to the most specific structures of music, and because also all that concerns acoustic systems, temperaments of tuning, the theoretical tonal systems, the intervals of genres and modes of music is the final concern of the exact sciences and can be the subject of descriptions and studies that can be quantified with precision.

CRITERIA FOR MEASURING SOUNDS AND INTERVALS

Numerical proportions on the string

General principles and the formation of the octave: 2/1

The reference to the exact sciences and to quantifiable values clearly implies recourse to some units of a precise nature of measurement to support a comparative objective approach. Since ancient times, the sounds and intervals between the sounds can be expressed on the vibrating string of a theoretical instrument, the monochord, or on the vibrating string of a real instrument, permitting a precise calibration of sounds. This was the case, within the Arabo-Islamic civilization, of the short-necked lute, ***al-'ūd,*** whose development and evolution are linked to this civilization, and more rarely of the long-necked lute, *ṭunbūr,* the hurdy-gurdy, *rabāba,* or other instruments. The sounds emitted by the string are expressed then in the form of numerical relations, i.e. a taut string from the nut on the left (on the edge of the neck) to the tailpiece on the right. If the whole of the chord vibrates, emitting for example a sound doh (C) 2, it is a 1/1 proportion. If, from the nut on the edge of the neck of the instrument, and on the left in the majority of diagrams, one places a finger of the left hand on the middle of the string whilst pressing on the fingerboard, and with the plectrum held in the right hand one plucks the right-hand part of the string, only this right half delimited by the pressing finger of the left hand and the tailpiece vibrates, whereas the left half of the string delimited by the pressing finger and the nut situated at the edge of the neck does not produce sound. We thus have a 2/1 proportion, 2 being the total length of the string and 1 the portion vibrating, and one obtains a sound of the high pitched octave of the precedent, for example a doh (C) 3. It is necessary in effect to know that, in all the systems, if the tension of the string remains constant, division by 2 of the length of the vibrating string multiplies the frequency and therefore the pitch of the emission by two, making it climb an octave. Conversely, multiplication by 2 of the length of the vibrating chord makes the sound lower by one octave.

Pythagorean system

If one presses the finger a third of the way along the string starting from the nut the two-thirds situated to the right vibrate under the plectrum; one has a 3/2 proportion, i.e. a perfect fifth, here for example a soh (G) 2. If one presses the finger a quarter of the length starting from the nut, the three-quarters situated to the right vibrate; one has a 4/3 proportion, i.e. a perfect fourth emitting here for example a fah (F) 2. In the Pythagorean system, the major second or major tone is defined by the difference between the fifth 3/2 and the fourth 4/3, i.e. 9/8. One therefore produces the first major tone by placing the pressing finger on a ninth of the length of the string starting from the nut; the eight-ninths situated to the right vibrate, and one therefore has the proportion 9/8, emitting here for example a ray (D) 2. The sum of two major tones, or ditone, defines the Pythagorean major third that one defines also by the sum of four-fifths (here for example doh-soh-ray-lah-mi) and by the numerical proportion 81/64, implying therefore a mental division of the string into eighty-one segments of which only sixty-four located to the right vibrate, which gives here for example a mi (E) 2. In the same Pythagorean system, the interval of rest obtained when from a fourth, 4/3, one subtracts a major third 81/64 is a 'limma' or small semi-tone defined by the proportion 256/243, which starting from the nut would give here for example a (D) 2 low flat ray. And the interval obtained when from a major tone 9/8 one subtracts a limma 256/243 is an 'apotome' or a large semi-tone defined by the proportion 2187/2048 which starting from the nut gives here for example a (C) 2 high sharp doh. The interval obtained when at an apotome 2187/2048 one subtracts a limma 256/243 is a Pythagorean comma defined by the proportion 531441/524288 (also defined by the difference between twelve fifths and seven octaves, or the difference between six tones and an octave). Finally the interval obtained when from a major tone 9/8 one subtracts a Pythagorean comma 531441/524288, or when one wants to situate a Pythagorean 'neutral second', or when one adds two limmas 256/243 (and that is why I call this proportion 'dilimma') is defined by the numerical proportion 65536/59049, which gives here for example starting from the nut, since the empty string is supposed to produce a doh (C) 2, a ray (D) less a comma which will be defined in the 'Arabesques' code of abbreviation by the abridged formula ray.d.2 (ray.2 flattened by a comma). This diminished Pythagorean third, or 'dilimma' or neutral second, has a value very close to that of the 'minor tone' of the harmonic system defined by the proportion 10/9. And although the acoustics expert must not confuse these mathematical definitions, the musician plays them more or less at the same value.

Harmonic systems (Aristoxenus, Zarlino, Delezenne etc.)

We have just seen the way in which a Pythagorean system can be calculated on the string of a monochord or played on the string of a lute or a violin by reference mainly to the cycles of fifths and to what can be extrapolated from them. This Pythagorean system is evidently the main system of musical acoustics and it remains extremely important in the Arabo-Islamic and the European world. Nevertheless, some other acoustic systems define other numerical proportions and therefore intervals and sounds of different pitch.

Thus one finds in the harmonic systems the same fifths 3/2, fourths 4/3, major tones 9/8, but one finds some new intervals: harmonic major third 5/4, minor third 6/5, major tone 9/8, minor tone 10/9, a sort of two-thirds of tone 27/25, a large semi-tone or pseudo-apotome 16/15, a small semitone or pseudo-limma 135/128, a minim semi-tone 25/24, a diesis 128/125, a syntonic comma 81/80, a diaschisma 2048/2025 etc. Between a Pythagorean system and a harmonic system there is therefore a host of differences at the level of pitch of sounds. But there are places where the two systems are in quasi-coincidence, as for example in the level of the limma (256/243 for 135/128), of the apotome (2187/2048 for 16/15), of the dilimma (65536/59049 for 10/9) or of the 'neutral third' (4a diminished 8192/6561 for 5/4).

Division of the string into forty aliquot segments (Eratosthenes)

Another acoustic system, which we associate with Eratosthenes and which the Arabs were using in the pre-Islamic era of the *Jāhiliyya,* consists of mentally dividing the string into forty aliquot segments. If, starting from the nut, the pressing finger delimits from the open string each possible fingering degree, one has the numerical proportions 40/40 open; by immobilizing a fortieth, 40/39; two fortieths, 40/38=20/19; three fortieths, 40/37; four fortieths or a tenth 40/36=20/18=10/9 equivalent to a minor harmonic tone; then 40/35=8/7 is worth a maxim tone; 40/34=20/17; 40/33; 80/64=40/32=20/16=10 /8=5/4, is worth a major harmonic third; 40/31; 40/30=4/3 is worth a perfect fourth; 40/29; 40/28=20/14=10/7 is worth an augmented harmonic fourth or tritone (three tones); 40/27 is worth a short fifth diminished by about one comma; 40/26 is worth a fifth augmented by about two commas and very near to the 'quinte du Loup'—'Wolfs 5a', 192/195—strongly discordant, which hindered European acoustics experts until the arrival of an even temperament at the beginning of the eighteenth century. One notes that this system of forty segments does not have the perfect fifth, and that it sometimes coincides with the harmonic system.

Linear measurements on the string

General principles

In the same way it is possible, on the string of the theoretical monochord or on the string of an instrument with a smooth fingerboard without frets, to define all the conceivable intervals extemporaneously by referring to the numerical proportions establishing the relation between the length of the open string from the nut to the tailpiece, a length called ‘diapason’, and the length of the vibrating portion delimited by the pressing finger by applying a mental division. This method which is concerned with numerical proportions is characteristic of ancient Greece, and it has been perpetuated to this day by innumerable musicologists and acoustics experts of the Arabo-Islamic world, or of other civilizations, most particularly the medievalists. The difficulty in this method resides in the fact that if there is an absence of great erudition, the announcement of a numerical proportion formed from complex numbers does not always give an immediate idea of where to press the fingers on the strings. It is necessary to make calculations in the majority of cases.

However, if one chooses a precise open length, and therefore a specified diapason between the nut and the tailpiece, all sound defined by a numerical proportion can be mentally situated on a string as a function of a linear measurement deduced from the previous data. At this point it is necessary to take into account the width of the pressing finger and almost unforeseeable factors, such as irregularities in the thickness of the strings or the styles of striking up, which will result in slight discrepancies between the the theoretical linear intersection and the practical position used to obtain a given sound. On the monochords, two strings are identically strung and with the same tension due to counterweights of the same mass. They have the same ‘diapason’, i.e. they are strung open between a nut and a tailpiece, or more often a bridge, evenly spaced out. And while one of the strings keeps a fixed diapason, the open diapason of the other is changed by the longitudinal displacement of its bridge. One can therefore shorten it at will, thus causing a controllable and measurable increase in its emission.

Linear measurements in the Pythagorean system

For reasons associated with the idea of facilitating the calculations, the strings of laboratory monochords are often a diapason of 1000 mm, i.e. 1 m. In this way it is easier to determine the locations to give to known intervals, for example to the principal Pythagorean intervals: octave 2/1, 500 mm; perfect fifth 3/2, 333.3 mm; perfect fourth 4/3, 250 mm; ditone or major third 81/64, 209.9 mm; augmented second 19683/16384, 167.6mm; minor third 32/27, 156.25mm; major second or major tone 9/8, 111.11 mm; apotome 2187/2048, 63.55mm; limma 256/243, 50.78mm; comma 531441/524288, 13.45 mm (all measurements start from the nut, of course).

On instruments destined for real playing, the precalculated numerical data are not as simple. On the long-necked lutes, such as the medieval **ṭunbūr** and its modern derivatives such as the *tanbūr* of Turkey, the strings of about one metre are fixed for the left hand, which defines the height of the pressing of large longitudinal displacements (except among left-handed musicians who use the right hand). On violins, whose strings are relatively short, the fingers of the left hand have to perform a quite tight and precise manner of playing in order to delimit the pitch of the sounds produced. On the short-necked lutes, such as the European lutes and the **ʿūds** of Arabo-Iranian-Turkish and similar music, practice led lute players to avoid very short strings, which make movement of the fingers of the left hand too tight, and very long strings, which make this movement too wide. The majority of the strings of *'ūds* have an open 'diapason' of 600 mm, sometimes more in the Maghreb, sometimes less on oriental *'ūds* endowed with prestigious signatures (Mannol, Onnik in Istanbul, 'Alī, Fāḍil in Baghdad) whose strings often have diapasons of 585 mm.

In order to give average values that are easy to calculate we shall indicate the positions to give to the most well-known Pythagorean intervals on an *'ūd* with 600 mm strings (all measurements are from the nut, of course): octave 2/1, 300 mm; perfect fifth 3/2, 200 mm; perfect fourth 4/3, 150 mm; major third (ditone) 81/64, 125.92 mm; augmented second 19683/16384, 100.56 mm; minor third 32/27, 93.75 mm; major second tone 9/8, 66.66 mm; apotome 2187/2048, 38.13 mm; limma 256/243, 30.47 mm; comma 531441/524288, 8.07 mm.

Linear measurements comparative with other systems

It is clear that the theoreticians should never give in to the temptation of confusing the different acoustic systems with one another. It is therefore essential to know the differences that separate the main traits of the different systems, not only in the theory but also in practice, on the strings, and more particularly on the strings of an *'ūd* with the 'diapason' of 600 mm between the nut and the tailpiece, a quite common 'diapason' on the *'ūd*.

All the octaves are the same with a proportion of 2/1 and therefore a position of finger pressing 300 mm from the nut. The fifths present slight differences; perfect fifths 3/2, 200 mm; equal tempered fifths 433/289, 199.53 mm (a comma 'schisma' 32805/32768, 0.676 mm to the nut, separates these two fifths). For the fourths, perfect fourths 4/3, 150 mm, but equal tempered fourths longer than the Pythagorean one, which is rare, 303/227, 150.49 mm (with again a difference of a comma schisma 32805/32768). For the thirds and seconds, in decreasing order: 3a major Pythagorean 81/64, 125.92 mm; 3a major even tempered 63/50, 123.80 mm; 3a major harmonic 80/64=5/4, 120 mm; 2a augmented Pythagorean 19683/16384, 100.56 mm; 3a minor harmonic 6/5, 100 mm; 2a augmented or 3a minor equal 44/37, 95.45 mm; 3a minor Pythagorean 32/27, 93.75 mm; 2a augmented harmonic 75/64, 88 mm; 2a major Pythagorean (major tone) 9/8, 66.66 mm; 2a major even tempered 449/400, 65.47 mm; minor harmonic tone 10/9, 60 mm; barely greater than the Pythagorean dilimma (3a diminished Pythagorean or 2a neutral Pythagorean) 65536/59049, 59.39 mm. For the semi-tones, the Pythagorean apotome 2187/2048, 38.13mm, is barely greater than the large harmonic semi-tone 16/15, 37.50 mm; the even tempered semi-tone 89/84, 33.70 mm; the small chromatic semi-tone 135/128, 31.11 mm, is barely greater than the Pythagorean limma 256/243, 30.47 mm. For the commas, we have: Pythagorean comma 531441/524288, 8.07 mm; Holderien comma $\sqrt[53]{2}$, 7.79 mm: syntonic or didymic comma of the harmonic systems 81/80, 7.41 mm; harmonic diaschisma 2048/2025, 6.74 mm; harmonic schisma 32805/32768, 0.676 mm. We thus observe the zones of confluence and the degrees to which the systems are intertwined.

Measurements of the pitch of sounds and of the intervals independent of the string and the system (hertz, savart, cent)

We have seen above that since antiquity the measurement of the pitch of sounds has been principally carried out on the string of the monochord. These sounds were defined by numerical proportions and they could also be expressed, knowing the length of the string or diapason, in precise linear measurements. But from now on sounds are produced independently of the string and even independently of the musical instrument. Units of definition or of measurement no less independent have therefore been created and used: the hertz (unit of frequency), the savart and the cent (units of measurement) and the Holderien comma, comma degree or 1/53 of an octave.

Inflections intended for modal music (non-tempered), code 'Arabesques'

The different ways of defining or measuring the pitch of a sound have been studied: numerical proportions and linear measurements, hertz, savarts, cents, Holderien commas etc. But for centuries the habit has been adopted of giving the names of notes to the degrees of a mode represented in the form of a gamut and on a Western stave of five lines and four spaces. As in the West one described only six and then seven names of notes, i.e. ut-re-mi-fa-sol-la-si to define an octave capable of having twelve virtual degrees; inflections elevating or lowering the note have to be used, which are the sharps and flats. This allows us already to distinguish the major doh (doh-ray-natural mi-fah-soh-lah-te-doh) from the minor doh (doh-ray-flat mi-fah-soh-flat lah-flat te-flat doh). But these conventional inflections are lacking in precision when it is a question of transcribing ancient music or music from outside Europe.

Certain authors have described systems of musical writing and staves intended for music of Arabic type since the thirteenth century. But musical writing seems essentially contemporary with the discovery of occidental musical staves by Middle Eastern people around the eighteenth century, and more particularly in the nineteenth century. As in this epoch, writing with its sharps and flats applied essentially to even temperament of twelve even semi-tones per octave, the Arabs and Iranians had to put half sharps and half flats to place the degrees apart from each other by intervals of three fourths of a tone, five fourths of a tone or seven fourths of a tone. Thus were born the *nuss*-sharps or *kar*-sharps and *nuss*-flats or *kar*-flats of the Arabs, and the *soris* and *korons* of the Iranians. The Turks, who had retained a Pythagorean comma system introduced in the thirteenth century by Ṣafī al-Dīn and improved more recently, have a code of alteration that is very precise but which does not allow for totally free transposition. The Arabic and Iranian inflections, however, are quite insufficient and led the Arabs and Iranians to imagine that their music could be defined and measured in fourths of a tone, which was only an approximation by compromise with the European musical staves.

It will be seen later that because the Arabo-Iranian-Turkish and similar modes are in general heptatonic and thus made up of seven degrees in the octave, a theoretical scale of sounds of twenty-four virtual fingering degrees per octave suffices to define, by selection of seven of these fingering degrees, a heptatonic musical mode. The presence of these twenty-four virtual fingering degrees per octave—there are only twelve in the West—led Middle Eastern people to think that the even temperament of twelve fingering degrees separated by twelve equal semi-tones, adopted in the eighteenth century in Europe, could have for the oriental equivalent a temperament almost equal to twenty-four fingering degrees separated by twenty-four fourths of equal tone (quarter tones).

For these reasons, the oriental code of inflections with semi-sharp, sharp, flat, semi-flat only constitutes an approximation that good musicians will have to interpret in an enriching way by re-establishing the intervals from a richer acoustic system. Certain learned musicians, such as the lute players of the school of Baghdad, or of Aleppo, did this.

The Turks use their own code, with over the length of the tone: comma, limma, apotome and dilimma. In the 1970s, a colloquium of musicologists held in Beirut and described by Ṣalāḥ al-Mahdī in his work *La Musique Arabe* (1972) wanted to enrich the usual code of inflection, but no 'etymological' explanation was given to justify the written reintroduction of these inflections.

In order to situate all the inflections with a precision of one comma, and therefore to authorize all transposition, I put together in 1978 a commatic code of inflections which I called 'Arabesques'. This code 'Arabesques' collates Arabic, Iranian and Turkish codes, the fourths of tone and the commas. Assigning one fourth of a tone to two Holderien commas, it takes up numerous signs already used on oriental staves and creates some new ones to situate the inflections concerning the nine commas of the major tone. The commatic code is given in Table 17.1. (See also Chabrier (1985).)

Table 17.1 J.-C.Chabrier. 'Arabesques' code of inflections, division of a tone into nine dots

0 natural	non altered diatonic degree. String nut, etc
1	tone up a bit by 1 Holderien, syntonic, Pythagorean comma tone down a bit by 8 H. commas, 1 dilimma or 1 minor tone
2	tone up a bit by 2 H. commas, 1 diesis 128/125 or quarter of a tone tone down a bit by 7 H, commas or 3/4 of a tone or (1125/1024; 12/11)
3	tone up a bit by 1 minime semitone, 3 c., or the equivalent (25/24) tone down a bit by 1 maxime semitone, 6c. or equivalent (27/25)
4	tone up a bit by 4 H. c., 1 limma or 1 little semitone (135/128) tone down a bit by 5 H. c., 1 apotome or 1 big semitone (16/15)
4.5	tone up a bit by 1 tempered equal semitone or 2 quarter tones tone down a bit by 1 tempered equal semitone or 2 quarter tones
5	tone up a bit by 5 H. c., 1 apotome or 1 big semitone (16/15) tone down a bit by 4 H. c., 1 limma or 1 little semitone (135/128)
6	tone up a bit by 6 H. c., or 1 maxime semitone or equiv. (27/25) tone down a bit by 3 H. c., or 1 minime semitone or equiv. (25/24)
7	tone up a bit by 7 H. c., 3 quarter tones or equiv. (1125/1024; 12/11) tone down a bit by 2 H. c., 1 diesis 128/125 or quarter of a tone
8	tone up a bit by 8 H. c., 1 dilimma or a minor tone tone down a bit by 1 H. c., syntonic, Pythagorean
9	natural non altered diatonic degree one tone higher than the previous

The decision to use the division of a tone into nine Holderien commas clearly allows inflections to be situated on the ninth of the tone and to distinguish the major Pythagorean thirds from the major harmonic thirds. But it is interesting to study, one by one, the nine gradations inside the length of the first tone to apprehend the coincidences or approximations with the degrees or intervals stemming from the diverse acoustic systems known across the world.

This code of inflections that I have used with success in all the analyses carried out since 1978 has turned out to be sufficiently precise to fulfil the functions for which it was intended.

Theoretical scale of virtual sounds 'tone system', code JCC of twenty-four virtual fingering degrees per octave

During the passage from the scale to the mode in even temperament in the West, twelve virtual fingering degrees per octave, by selection of seven degrees per octave, allow a definition of the heptatonic mode. In Arabic and similar music, a theoretical scale of twenty-four virtual fingering degrees per octave, also by selection of seven degrees per octave, allows the definition of a heptatonic mode. It is therefore useful to set up a nomenclature of these twenty-four fingering degrees equipped with indications in letters or figures which, when a style is being defined, will eliminate all reference to the pitch or the names of notes and which will be quite flexible for adapting to several neighbouring temperaments. A 'code JCC' has therefore been established which favours transpositions (Table 17.2).

Comparison of criteria of measurement and a gamut

A certain number of elements or units of measurement which can give information about the pitch of sounds, about the intervals that separate two or more sounds and about the organization of sounds in the framework of an acoustic system have therefore been identified. These elements, i.e. the numerical proportions, the linear measurements on strings deduced from their proportions, the units of measurement such as the hertz, the savart or the cent (these last ones will only be mentioned with the abbreviation '°'), the comma degrees represented by the Holderien comma (at a rate of nine per tone and fifty-three per octave), the code of inflections 'Arabesques' (dividing the tone into nine marks like the Holderien commas) and the nomenclature called 'code JCC' of twenty-four fingering degrees per octave, are going to permit us, at an initial stage, to explore theoretical or virtual possibilities of the scale of sounds.

Table 17.2 Nomenclature 'code JCC' of the twenty-four fingering degrees/octave

17 dd system	*note in C*	*code JCC*	*quarters numbers*	*commas numbers*	*transposition towards the Pythagorean system*
+	C	0.A	0	0	
		0+A		1	Comma

		1.B	1	4 Limma
+		2.C	2	5 Apotome
+		3.D	3	8 Dilimma. neutral 2a. 3ad
+	D	4.E	4	9 major 2a. major tone
		4+E		10 major 2a+comma
		5.F	5	13 minor 3a
+		6.G	6	14 augmented 2a
+		7.H	7	17 neutral 3a. diminished 4a
+	E	8.1	8	18 major 3a—ditone
		8+I		19 major 3a+comma
		9.J	9	21 neutral 4a
+	F	10.K	10	22 perfect 4a
		10+K		23 augmented 3a. 4a j+c
		11.L	11	26 diminished 5a
+		12.M	12	27 augmented 4a. tritone
+		13.N	13	30 diminished 6a. neutral 5a
+	G	14.0	14	31 perfect 5a
		14+0		32 perfect 5a+comma
		15.P	15	35 minor 6a
+		16.Q	16	36 augmented 5a
+		17.R	17	39 diminished 7a. neutral 6a
+	A	18.S	18	40 major 6a
		18+S		41 major 6a+comma
		19-T		43 minor 7a—comma
		19.T	19	44 minor 7a
+		20.U	20	45 augmented 6a
+		21.V	21	48 diminished 8a. neutral 7a
+	B	22.X	22	49 major 7a
		23.Y	23	52 diminished 9a. neutral 8a
	C	24.Z	24	53 perfect 8a

First we present a chromatic Pythagorean scale such as could be played on a lute-*ʿūd* whose strings would have a diapason of 600 mm. To simplify, we assume that the open string gives a doh (C) 2. Table 17.3 will be organized in the following fashion.

First column: name of notes from doh to doh with inflections according to the 'Arabesques' code

Second column: counted starting from the nut, the distance at which it is necessary to apply the left finger to obtain the designated sound

Third column: numerical proportions with the diapason of the string

Table 17.3 Realisation of a Pythagorean chromatic scale

notes in C	*mm/600 /string*	*numerical ratio diapason*	*cents 1200/oct*	*Holder 53/oct*	*code JCC*	*abr*	*name of the interval*
C.♮	open string	1/1	0	0	0.A	…	
C .+	8.07	531441/524288	23.5	1.04	0+A	+	Pythagorean comma
D.♭	30.47	256/243	90. 2	4	1.B	2am	limma. minor second
C.♯	38.13	2187/2048	113. 7	5	2.C		apotome
D.d	59.39	65536/59049	180.5	8	3.D	2an	dilimma. neutral second. 3ad
D.♮	66.66	9/8	203. 9	9	4.E	2aM	major tone. major second
D .+	73.8	4782969/4194304	228	10	4+E	2a+	major second+comma
E.♭	93.75	32/27	294.1	13	5.F	3am	minor third
D.♯	100.56	19683/16384	317.6	14	6.G	2aA	augmented second
E.d	119.46	8182/6561	384. 4	17	7.H	3an	neutral third. diminished 4a
E.♮	125.92	81/64	407. 8	18	8.1	3aM	major third+ditone
E.+	131.9	43046721/33554432	432	19	8+1	3a+	major third+comma
F.d	143.9	2097152/1594323	474	21	9.J	4an	neutral fourth
F.♮	150	4/3	498	22	10.K	4aJ	perfect fourth
F.+	156	177147/131072	521. 5	23	10+K	3aA	augmented third. 4a+comma
G.♭	172.85	1024/729	588. 3	26	11.L	5ad	diminished fifth
F.♯	178.6	729/512	611. 7	27	12.M	4aA	augmented fourth. tritone

G .d	194.5	262144/177147	678. 5	30	13.N	5an	diminished sixth. neutral 5a
G.♮	200	3/2	702	31	14.0	5aJ	perfect fifth
G.+	205.4	1594323/1048576	726	32	14+0	5a+	perfect fifth+comma
A.♭	220.3	128/81	792. 2	35	15.P	6am	minor sixth
G.♯	225.4	6561/4096	815. 6	36	16.Q	5aA	augmented fifth
A.d	239.5	32768/19683	882.4	39	17.R	6an	diminished seventh. neutral 6a
A.♮	244.4	27/16	905.9	40	18.S	6aM	major sixth
A.+	249.1	14348907/8388608	930	41	18+S	6a+	major sixth+comma
A.3	257.8	8388608/4782969	972	43	19-T		minor seventh–comma
B.♭	262.5	16/9	996.1	44	19.T	7am	minor seventh
A.♯	267	59049/32768	1019. 6	45	20.U	6aA	augmented seventh
B.d	279.6	4096/2187	1086.3	48	21.V	7an	diminished octave. neutral 7a
B.♮	284	243/128	1109. 8	49	22.X	7aM	major seventh
C.d	295.9	1048576/531441	1176.5	52	23.Y	8an	diminished ninth. neutral 8a
C.♮	300	2/1	1200	53	24.Z	8aJ	perfect octave

Fourth column: interval in cents with the open string vibrating
Fifth column: interval in Holderien commas
Sixth column: JCC code
Seventh column: abbreviation of the name of the interval Eighth column: complete name of the Pythagorean interval

Note that it is a question here of a Pythagorean gamut comprising certain extrapolations such as those ensuing from the interpretation of the Pythagorean system by the Middle Eastern musicologist Şafī al-Dīn al-Urmawī al-Baghdādī in the thirteenth century, and by the enrichment which occurred in Turkey. This gamut corresponds more or less to that which a very cultured Turkish or Iraki lute player could have interpreted in the twentieth century.

STAGES OF ARABIC MUSICAL THEORY

We will assume henceforth that the essential data of musical acoustics are known. At this level, the framework of the octave is no longer unknown and what is going to be revealed about the evolution of musical theories conceived or described in the Arabo-Islamic civilization will fit into an explored domain.

From the first centuries of Islam the scholars tried hard to apply a scientific status to the empirical intervals such as the neutral second, situated between the semi-tone and the whole tone, i.e. approximately at three fourths of a tone, or the neutral third, situated between the minor third and the major third, i.e. approximately at seven fourths of a tone.

It is now necessary to begin a description of the systems which followed one another in the matter of Arabic music.

We do not need numerous indices to explain the musical theories which could have been recognized or observed by the Arabs in the time of the *Jāhiliyya* before Islam. Nevertheless we can consult one of the most famous musicologists of the Arabo-Islamic world, al-Fārābī, who, in the tenth century, in the *Kitāb al-Mūsīqī al-Kabīr (Great Book of Music),* described in Book 2, Essay 2, a musical instrument called the *ṭunbūr* from Baghdad in precise terms of which a translation is found in *La Musique arabe*.[1]

Al-Fārābī describes the usage, on the long-necked lute, of an acoustic system that he attributes to pre-Islamic musicians and that consists of placing five frets, starting from the nut, equidistant separated one from the others by aliquot segments of which each equates to a fortieth of diapason of the string. If for example we have a string of 600 mm diapason these five frets are respectively a distance from the nut of 15, 30, 45, 60 and 75 mm. These frets are described as 'pagan' and are dedicated, writes al-Fārābī, to the playing of 'pagan' airs. By 'pagan' he means here pre-Islamic. Controlled by four fingerings, these five frets are distributed between the nut and an eighth (8/7; 75 mm for 600 mm) of the diapason of the string. Thus one only plays on a length of string barely passing the major tone: if one accepts this description, one can deduce that, whatever the interval of tuning between two lines of consecutive strings, the playing of such instruments is limited to very primitive airs. Furthermore, as the frets are separated by geometrically equidistant segments, the intervals that they determine are acoustically uneven. This led al-Fārābī to suggest a spacing of frets as a function of geometrically decreasing intervals in order to attain acoustically constant intervals.

In the continuation of his exposition al-Fārābī mentions (or imagines) the existence of three supplementary frets distributed between the eighth (8/7) and the fifth (5/4) of the diapason, i.e. at distances from the nut of 90, 105 and 120 mm respectively. He even envisages eventually the addition of two new supplementary frets, which, by giving the positions 40/31, 135 mm, and 40/30=4/3, 150 mm (a quarter of the diapason), would permit attainment of the fourth. He also explains that in disposing these supplementary frets as a function of decreasing geometric intervals, one could obtain acoustically equal intervals according to a system which he called 'feminine'.

Al-Fārābī calls to mind again the different possibilities of harmony between two or three strings of the *ṭunbūr* of Baghdad. Thus one could tune them in unison (which gives the instrument a very close register), with an interval of a limma (which may seem inadequate) or, better still according to al-Fārābī, with an interval of a fourth (which could at least give the instrument acceptable melodic possibilities).[2]

By mentioning this description by al-Fārābī of a pre-Islamic acoustic system applied in the tenth century on the *ṭunbūr* of Baghdad by certain musicians, and improvements that one could bring there, a number of recent authors have believed that this system can be defined as characteristic of the pre-Islamic Arabic civilization. This is the case with Kosegarten, Farmer and Barkechli.[3]

But such an affirmation is much more imprudent than the fact that dividing a string of an instrument into forty aliquot segments, and therefore the first octave into twenty aliquot segments, corresponds to an ancient process that is described in Eratosthenes.[4] This process does not therefore seem more specifically to apply to Arabs than to all people.

Whether it is specifically Arabic or not this acoustic system of division of the string into forty aliquot segments merits attention. And whereas al-Fārābī limits his description to a length of fifth, we are going to study it here on the whole of an octave.

Table 17.4 gives from left to right:

the number of millimetres starting from nut for a string of 600 mm
the numerical proportion and its eventual reduction
the value in cents in a ratio of 1200 cents per octave
the value in Holderien commas in the ratio of 53 holders per octave
the definition as a function of the JCC code (twenty-four divisions per octave)
equivalence or comment

The last three rubrics are not always mentioned.

Although this system is very archaic and has not survived to the present day, it is perhaps because it is a little deficient. But it is interesting to

Table 17.4 A. Ancient system string division into 40 aliquotic sections

mm/ 600 string	*ratio*	*cents 1200/oct*	*Holder 53/oct*	*code JCC*	*equivalence or commentary (cf comparative table, register "D")*
15	40/39	44	2–	1.B	infra-quarter tone, Eratosthenean diesis
30	40/38=20/19	89	4–	2.C	Pythagorean infra-limma, Delezenne's infra-chromate.
45	40/37	135	6+	3.D	Zarlino's supra-diat. (27/25), Ibn Sīnā's infra neutral 2a 13/12
60	40/36=10/9	182	8	4.E	precise harmonic minor tone
75	40/35=8/7	231	10	5.F	maxime tone, big tone, cf Iran XXC
90	40/34=20/17	281	12.4	6.G	between harmonic aug. 2a & Pyth. minor 3a
105	40/33		15–	7.H	between Pyth. aug. 2a & low neutral 3a (39/32) according to Ibn Sīnā
120	40/32=5/4	386	17	8.1	precise harmonic major third
135	40/31		19.5	9.J	diminished fourth

150	40/30=4/3	498	22	10.K	perfect fourth
165	40/29		24.5	11.L	Zarlino's infra-aug. 4a (25/18, 570 c, 168 mm, 25 h)
180	40/28=10/7	617	27+	12.M	precise harmonic tritone (aug. harmonic 4a)
195	40/27		30	13.N	short fifth of inequal temperament europ
210	40/26=20/13		33	14.0	Loup's supra-fifth (192/125), inequal temp.
225	40/25=8/5	814	36	15.P	precise harmonic minor sixth
240	40/24=5/3	884	39	16.Q	precise harmonic major sixth
255	40/23		42	17.R	augmented supra-sixth of Zarlino (125/72)
270	40/22=20/11		46	18.S	Zarlino's supra minor 7a; Pyth. supraaug 6a
285	40/21		49	19.T	Pythagorean supra major 7a (243/128)
300	40/20=2/1	1200	53	20.U	octave

Table 17.5

30mm	89°	40/38=20/19	infra limma, Delezenne's infra-chromate
60 mm	182°	40/36=10/9	precise harmonic minor tone
90mm	231°	40/35=8/7	maxime tone, big tone
120mm	386°	40/32=5/4	precise harmonic major third
150mm	498°	40/30=4/3	perfect precise fourth
180 mm	617°	40/28=10/7	precise harmonic tritone (4aAH)
195 mm	680°	40/27	short fifth
210 mm		40/26=20/13	Loup's supra-fifth (192/125)
225 mm	814°	40/25=8/5	precise harmonic minor sixth
240 mm	884°	40/24=5/3	precise harmonic major sixth
255 mm		40/23	Zarlino's supra-augmented sixth
285 mm		40/21	Pythagorean supra major seventh
300 mm		40/20=2/1	octave

discover that it enters at the level of certain fingering degrees, and more particularly of the harmonic system, i.e. more precisely at the levels given in Table 17.5.

One notes the absence of major tone and perfect fifth. But this system contains fingering degrees comparable to a neutral second and a neutral third that one will rediscover, not without fluctuations of pitch, in all the subsequent acoustic systems of this music.

Acoustic systems successively described from the blossoming of Islam to its decline

The Pythagorean system in Islam

al-Mawṣilī (*ʿūd*, ninth century)
al-Kindī (*ʿūd*, ninth century)
al-Munajjim (*ʿūd*, tenth century)
al-Fārābī (harp, tenth century)
See Table 17.6.

The musical tendencies of the first centuries of Islam are known through the treatises of al-Kindī (ninth century) and of al-Munajjim (tenth century) and by the translations or commentaries of great oriental musicologists of the twentieth century such as Rouanet, d'Erlanger or Farmer.[5]

According to al-Munajjim, Isḥāq al-Mawṣilī, a lutenist from Baghdad who was well known to the court of the Abbasid Caliphs, a lawyer and man of culture, partisan of musical classicism, applied the Pythagorean theories of the 'Ancients' (the Greeks) although declaring not to know them. A

Table 17.6 Pythagorean system of the first centuries in Islam (Al-Mawṣilī, Al-Kindī)

mm/ 600 string	*ratio*	*cents 1200/ oct*	*commas 153/ oct*	*fingering degree*	*commentary, equivalence (cf comparative table; register 1)*
30.47	256/243	90°2	4	(neighbour-forefinger)	limma (not on forefinger)
38.13	2187/2048	113°7	5	(neighbour-forefinger)	apotome (not on forefinger)
66.66	9/8	203°9	9	forefinger	major tone
93.75	32/27	294°1	13	"ancient" middle finger	minor third
125.92	81/64	407°8	18	ring finger	major third
150	4/3	498°0	22	little finger	perfect fourth
178.6	729/512	611°7	27	shift	augmented tritone 4a
200	3/2	702°0	31	(next string)	perfect fifth

treatise of al-Kindī gives a description of the finger board of the lute-*ʿūd* from which one can deduce that the advocated system is Pythagorean.[6]

In this era, musical theory seems to be absolutely indissoluble from an experimentation of sounds and intervals on the string of a monochord, which is in general replaced by the finger board of a short-necked lute, *ʿūd*, that one tunes by fourths. One can thus apply the seven fingering degrees of a heptatonic musical style on two consecutive lines of strings with the help of four fingers on the left hand. As one only plays on a fourth per range of string, it is necessary to note that the response to the octave of the most low-pitched degree (i.e. the eighth degree) is reached on the second range through the technique called 'shift' which consists in placing the left hand along the neck in the direction of the sound-box of the *ʿūd* (and even in front of the sound-box near the tailpiece in the most subtle shifts). One thus obtains the response to the sharp octave of the low-pitched degree played on the first empty range on placing the finger on the fifth of the following range.

This method, which consists in studying an acoustic system on the finger board of the lute-*ʿūd*, is called the theory of 'fingerings' (*aṣābiʿ*) and it defines in this epoque and on the *ʿūd* eight different musical styles which have been described by al-Iṣfahānī in the tenth century, in the *Kitāb al-aghānī,* the *Book of Songs,* widely commented on in the twentieth century.[7]

It is a simple Pythagorean system. It does not introduce any fingering degree defining intervals of an exotic type such as neutral seconds or neutral thirds. It describes a limma, an apotome, a major tone, a minor third, a major third, a perfect fourth, a tritone[8] and a perfect fifth on the following range.

Para-Pythagorean system of Zalzal (eighth century): empirical longitudinal divisions of the string

If we believe lute players like al-Mawṣilī, narrators such as al-Iṣfahānī and al-Munajjim and theorists like al-Kindī and al-Fārābī, it seems that this era is characterized by the coexistence of several acoustic systems. There was already a coexistence between a system based on the division of the string into forty aliquot segments by the placing of equidistant frets, and a system of Pythagorean type defining intervals like the limma, apotome, major tone or major second, minor third, the ditone or major third, the fourth, the tritone or augmented fourth, and the fifth. We saw above the exact interferences (perfect fourth, octave) or approximate interferences (limma, seventh, major) between these two systems.

We cannot know precisely whether, in addition to these two systems, other theories were applied in this era. But we note that at the end of the eighth century a lute player from Baghdad called Manṣūr Zalzal, brother-in-law of Ibrāhīm al-Mawṣilī and therefore uncle of Isḥāq al-Mawṣilī, imagined or introduced, in addition to the Pythagorean system, some fingering degrees of which he defined the positions to start from universal Pythagorean indicators. The essential principle of the calculations of Zalzal rest on the division of a given distance delimited by two fingering degrees into two equal parts, with adoption of the point equidistant from the two extremities to define the location of fingering positions before they are introduced into the system. It is therefore a method resting on the equidistance by longitudinal divisions on the string.

Already, if we believe Farmer, the ‘medius ancien’ (ancient middle finger) or Pythagorean minor third (32/27, 294°1, 13 h, 93.75 mm), normally situated on a major tone below the fourth (see the Pythagorean system on the fourth line of Table 17.7), is judged to be too high or badly calculated. The musicians of the time divided into two equal parts the distance between the fingering of the index finger, major Pythagorean second (9/8, 203°9; 9 h, 66.66 mm) and the fingering of the third finger, ditone of the major Pythagorean third (81/64, 407°8; 18 h, 125.92 mm), and thus defined a fingering of the middle finger controlling a new minor third, higher pitched, which they called Persian minor third (81/68, 303°, 13.4 h, 96.29 mm). By this principle of equidistance one has therefore raised the minor third by nine cents, i.e. almost a semi-comma (on the comparative table of the divisions of the fifth, this calculation is represented by two arrows marked by two oblique bars).

It is now a matter of defining by equidistance the fingering of a neutral third situated at mid-distance from the Persian minor third (81/68, 303°, 13.4h, 96.29mm) and the Pythagorean major third or ditone (81/64, 407°8, 18 h, 125.92 mm), which gives the middle finger of

Table 17.7 Longitudinal empirical divisions of a string (VIIIthc)

mm/ 600 string	*link*	*hundred Farmer*	*commas*	*fingering degree*	*commentary, equivalence (cf comparative table; register “I”)*
30.47	256/243	90°2	4	ancient neighbour	limma (persistent)
48.15	162/149	145°0	6.4	Persian neighbour	infra 3 quarters of a tone
55.55	54/49	168°0	7.4	Zalzal neighbour	infra minor tone
66.66	9/8	203°9	9	forefinger	Pythagorean major tone
93.75	32/27	94°1	13	ancient middle finger	Pythagorean minor 3a
96.3	81/68	303°0	13.4	Persian middle finger	supra minor 3a
111.11	27/22	355°0	15.7	Zalzal middle finger	neutral 3a
125.92	81/64	207°8	18	ring finger	Pythagorean major 3a
150	4/3	498°0	22	little finger	perfect 4a

Zalzal or neutral third of Zalzal (27/22, 355°, 15.7 h, 111.11 mm) (on the comparative table of the divisions of the fifth, this calculation is represented by two arrows marked with one oblique bar).

From these two new indicators two other derived fingering degrees have been calculated:

1 At mid-distance from the nut and the Persian minor third (81/68, 303°, 13.4 h, 96.29 mm) a neighbour of the index called the Persian second (162/149, 145°, 6.4 h, 48.15 mm), barely inferior to three fourths of a tone (on the comparative table of divisions of the fifth, this calculation is represented by two arrows marked with three oblique bars);
2 At mid-distance from the nut and the middle finger of Zalzal or neutral third of Zalzal (27/22, 355°, 17.7 h; 111.11 mm), a neighbour of the index called the neutral second of Zalzal (54/49, 168°, 7.4 h, 55.55 mm), slightly superior to three quarters of a tone and slightly inferior to a minor tone or to a dilimma (on the comparative table of divisions of the fifth, this calculation is represented by two arrows marked with four oblique bars).

The introduction of scholarly theories of fingering degrees called neutrals therefore date from the time of Manṣūr Zalzal, the fingering degrees being probably borrowed from native practices (the term neutral is a recent term), i.e. at the middle finger the neutral third of Zalzal, then two neutral seconds or neighbours of the index, the Zalzalian and the Persian.

Pythagorean systems of al-Fārābī (tenth century)

Among all the scholars of Islam, al-Fārābī (died in Damascus in 339/950) occupies an eminent place but we here only consider the musicological aspect. In this field, he is the author of *Kitāb al-mūsīqī al-kabīr (Great Book of Music),* of which several complete copies have permitted a critical analysis.[9]

From the introduction, al-Fārābī refers to the 'Ancients' who are evidently the Greeks. He is therefore going to compile a classic treatise. He defines music as being capable of stimulating amusement, the imagination and the passions; but he considers it inferior, as regards its impact, to poetry which conveys words. Then he studies the question of intervals and of eight 'genres' among which he describes a major genre, a 'neutral' genre and a minor genre.[10]

Book 1 is devoted to 'Elements of the science of musical composition'. He brings up the question of intervals which he has only imperfect command of, taking into account the knowledge of his time. He experiences difficulties dividing the major tone, only arriving at approximations through geometric longitudinal divisions which musically speaking are imprecise. In this way he describes a 'quarter of a tone or interval of slacking' corresponding to the division of the major Pythagorean tone (66, 66 mm, 9/8, 203°9, 9 h) into two geometrically equal semi-tones (the first, 33.33mm, 18/17, 98°, 4.33 h) or into four geometrically equal quarters of a tone (the first equal to 16.66mm, 36/35, 49°, 2.17h) (see Table 17.8). He also proposes two semi-tones of unequal proportions, superperfect or 'epimeres', i.e. 18/17 and 17/16.[11]

As regards the description of notes and styles, he puts them back into Greek terminology. His study of rhythm does not contain innovation. But he describes the manner in which one can construct a monochord capable of calibrating the sounds and measuring the intervals.[12]

Book 2 of the *Great Book of Music* by al-Fārābī is devoted to the instruments which are considered as a means of experimental control of musical theory. The first essay deals with the lute-*ʿūd*, its ‘fingerboard’ (frets or fingering degrees), its scale and its chords. One finds here the essential elements of the steps of al-Fārābī in the direction of musical science. Far from behaving as an innovator or a reformer, he realizes an encyclopedic approach in the context of which he expounds all the theories and uses of which he has knowledge. Thus one sees the description of a fourth given in Table 17.9.

As has been said above, the description of the musical scale of the lute-*ʿūd* by al-Fārābī has evidently been developed by an encyclopedic process, i.e. he lists and situates all the virtual fingering degrees of the theoretical musical scale, for example a quarter

Table 17.8 al-Fārābī. Notes. Intervals. *ʿūd*. Course of a fourth. 10 theoretical fingering degrees

abbr JCC	*mm/600*	*numerical ratio order in four*		*hundreds 1200/oct*	*Holder 53/oct*	*code JCC*	*name of the interval calculation on monostring or ʿūd*
						O.A	open string nut
1/4.T	16.66		36/35	49°0	2.17	1.B	equidistant quarter of a tone (tone divided into 4 equal sections)
2amp	30.47	1.	256/243	90°2	4–	1.B	Pyth. neighbour-forefinger limma (ditone substracted to 4a)
1/2.T	33.33	2.	18/17	98°0	4.33	2.C	semitone; neighbour-forefinger (equidistant from nut-major tone)
1aAp	38.13		2187/2048	113°7	5+	2.C	Pyth. apotome; neighbour forefinger (limma substracted to major tone)
2anf	48.15	3.	162/149	145°0	6.41	3.D	neutral persian 2a; neighbour-forefinger (equidistant nut-persian middle finger)
3/4.T	50		12/11	151°0	6.68	3.D	3 quarters of a tone; replica Zalzal middle finger on 1st string
2anz	55.55	4.	54/49	168°0	7.43	3.D	neutral Zalzal 2a; neighbour-forefinger (equidistant nut-Zalzal middle finger)
2aMp	66.66	5.	9/8	203°9	9	4.E	major 2a, major tone p.; forefinger 1/9 string; (fifth minus fourth)

3amp	93.75	6.	32/27	294°1	13	5.F	Pyth. minor 3a; neighbour-middle finger (a tone substracted to fourth)
3amf	96.29	7.	81/68	303°0	13.4	5.F	Zalzal minor 3a; persian middle finger (equidistant major tone-ditone)
2aAp	100.56		19683/16384	317°6	14+	6.G	Pyth. augmented 2a; Zalzal middle finger (limma substracted to ditone low)
3anz	111.11	8.	27/22	355°0	15.7	7.H	Zalzal neutral 3a; Zalzal middle finger (equidistant persian middle finger-ditone)
3aMp	125.92	9.	81/64	407°8	18	8.1	major 3a p.; Pythagorean ditone (limma substracted to fourth)
4ajp	150	10.	4/3	498°0	22	10.K	perfect 4a; little finger; 1/4 string (fifth substracted to octave)

Table 17.9 al-Fārābī. Notes. Intervals. 'ūd. course of a fourth. Fingering degrees

--- *quarters of a tone*

- quarter of a tone (1/4 linear of major tone: 16.66mm; 36/35; 49°; 2.17 h)

--- *forefinger's neighbours, semitones*

1- Pythagorean limma (4a minus ditone: 30.47 mm; 256/243; 90°2; 4 h)

2- semitone (halfway nut-major tone: 33.33 mm; 18/17; 98°; 4.33 h)

- Pythagorean apotome (major tone minus limma: 38.13 mm; 2187/2048; 113°7; 5 h)

--- *forefinger's neighbours, neutral seconds*

3- persian neutral 2a (halfway nut-persan middle finger: 48.15mm; 162/149; 145°; 6.41 h)

- 3 quarters of a tone (Zalzal's replica on first string: 50mm; 12/11; 151°; 6.68 h)

4- Zalzal's neutral 2a (halfway nut-Zalzal's middle finger 55.55 mm; 54/49; 168°; 7.43 h)

--- *forefinger, major second, major tone*

5- Pythagorean major 2a (5a minus 4a: 66.66 mm; 9/8; 203°; 9; 9 h)

--- *middle finger, minor thirds, augmented seconds, neutral thirds*

6- persian minor 3a, neighbour middle finger (4a minus tone 93.75 mm; 32/27; 294°1; 13 h)

7- persian minor 3a, Zalzal's persan middle finger (halfway ton-ditone 96.29mm; 81/68; 303°; 13.4 h)

- pers. augmented 2a, low Zalzal's middle finger, ditone minus limma 100.56 mm; 19683/16384; 317°6; 14 h)

8- Zalzal's neutral 3a, Zalzal's middle finger (halfway persian middle fingerditone: 111.11 mm; 27/22; 355°; 15.7h)

--- *ring finger, major third, ditone*

9- Pythagorean major 3a, ditone (fourth minus limma: 125.92 mm; 81/64; 407°8; 18 h)

--- *little finger, fourth*

10- Pythagorean perfect 4a (string fourth, octave minus fifth: 150mm; 4/3; 498°; 22 h)

---

of a tone, three semi-tones, three neutral seconds, a major second, four middle fingers (minor thirds, augmented seconds, neutral thirds), a major third and a perfect fourth. This makes fourteen fingering degrees to the fourth, and therefore several systems.[13]

But already, al-Fārābī limits the fingering degrees to ten per fourth:

> If we count the notes supplied by all the frets about which we have just spoken, plus those rendered by the strings in all their length, we find that each string produces ten notes.[14]

We can therefore picture that al-Fārābī divides the fourth into ten intervals.

Moreover, a given musical style only mobilizes four degrees per fourth and seven per octave. It only includes one second, one third, one fourth etc., and it must operate a selection among the theoretical fingering degrees, a selection which will only develop in the case of modulation.

Al-Fārābī is very clear about the distinction that it is necessary to make between the theoretical scale and the fingering degrees to select for playing:

> The ties that we have listed are almost all those which one ordinarily uses on the lute. One does not meet them all together, however, on the same instrument. There are those which are indispensable for the playing of the lute and are used by all musicians. These are the fret of the *index finger,* that of the *ring finger* and that of the *little finger,* and one of those between them which are placed between the fret of the index finger and that of the ring finger and are all termed the *frets of the middle finger,* this will be for some the *middle finger of Zalzal,* for others the *Persian middle finger* and for others still the fret which we have called the *neighbour of the middle finger.*
>
> As for the frets termed the *neighbours of the index finger,* certain musicians reject them and do not use them. Others use one of the *middle finger* frets and employ with this the *neighbour of the middle finger* that they consider good as such and not as a middle finger fret, but they do not employ any of those called *neighbours of the index;* others still use at one and the same time one of the two frets of the *middle finger,* the *neighbour of the middle finger* and one of the frets called the *neighbours of the index finger,* i.e. that which is separated by an interval of a limma of the fret of the index finger.[15]

Later, influenced by the Greeks, al-Fārābī recommends resort to a fifth string or to the practice of shift to enable the playing of the 'perfect group', i.e. the double octave.[16]

In Essay 2 of Book 2, al-Fārābī describes other instruments:

> The *ṭunbūr* from Baghdad, the system for which we have described above with division of the string into forty aliquot segments, dividing the fourth equally into ten musically unequal segments.[17]
>
> The *ṭunbūr* from Khurāsān: al-Fārābī recommends a system of the Pythagoro-commatic type with division of the octave into fifth, fourth, tone, dilimma, limma and Pythagorean comma. The division of the tone into two limmas and a comma, totally Pythagorean, foreshadows a system that will be found again in the thirteenth century on the *'ūd* with the treatise of **Ṣafī al-Dīn.** This system will be included later with that of **Ṣafī al-Dīn.**[18]
>
> The flutes: al-Fārābī studies the proportion between the measurements of flutes and the sounds emitted, the manner of arranging the openings and the scales of the sounds on the flutes.[19]
>
> The *rabāba:* al-Fārābī recommends in a quite unexpected manner a system of the paraharmonic type exploiting the following intervals:
>
> perfect fifth 3/2; tritone of 'Zarlino' 45/32;[20] perfect fourth 4/3; major Pythagorean third 81/64; major harmonic third 5/4; minor harmonic third 6/5; major Pythagorean tone 9/8; minor harmonic tone 10/9; pseudo apotome 16/15; pseudo limma 135/128; Pythagorean limma 256/243. This original system will be studied later.[21]
>
> The harps: al-Fārābī describes diverse intervals on the harps including an augmented 4a or Pythagorean tritone 178.6 mm, 729/512, 611°7, 27 h. It divides the fourth into two superperfect intervals and well-known consonants of respective proportions 8/7 and 7/6. It has already been seen that the interval 8/7=40/35 was a maxime tone present in the system of the division of the string into forty aliquot parts.[22]

The *Great Book of Music* by al-Fārābī ends with Book 3 devoted to 'musical composition' such as can be applied to instruments and the human voice, and to that which the human voice conveys of literary and poetic forms, or again to the stimulation of the senses and the soul, the stimulation of the soul being judged the most desirable. It is interesting to note that al-Fārābī goes back over what he had already affirmed in the introduction, i.e. that he considers vocal music to be superior to instrumental music.

The work of al-Fārābī therefore seems fundamental in the history of Arab music. Not that he has invented a system, but because he describes with encyclopaedic erudition all that is understood around him, what the Greeks did and what was done before Islam. We shall find elsewhere his writings in the study of the subsequent stages of the evolution of Arab and similar music.

Table 17.10 Ibn Sīnā-Avicenna (XIthc). Notes **'ūd.** Course of a fourth. Theoretical fingering degrees

abbr JCC	*mm /600*	*numerical ratio*	*cents 1200/oct*	*Holder 53/oct*	*code JCC*	*name of the interval calculation on monostring*
	0				O.A	open string nut
2am	37.36	273/256	112°0	5	2.C	ps. big sem. t. H; ps. apot.P; "Head" (maxime tone (8/7) under neutral 3a (39/32))
2an	46.15	13/12	139°0	6.15	3.D	neutral 2a (low); adjunct Zalzal's middle finger (major tone (9/8) under neutral 3a (39/32))
2aMp	66.66	9/8	203°9	9	4.E	major 2a p.; major tone p.; forefinger, 1/9 string (fifth minus fourth)
3amp	93.75	32/27	294° 1	13	5.F	Pyth. minor 3a; old middle finger (major tone under fourth)
3anz	107.69	39/32	343°0	15.17	7.H	Zalzal neutral 3a (low); Zalzal middle finger (equidistant forefinger-little finger)
3aMp	125.92	81/64	407°8	18	8.1	major 3a p.; Pythagorean ditone (fourth minus limma; ring finger)
4ajp	150	4/3	498º0	22	10.K	perfect 4a; 1/4 string; little finger (octave minus fifth)

Para-Pythagorean system of Ibn Sīnā (370–428/980–1087)[23]

See Table 17.10. In the eleventh century, Ibn Sīnā made an essential contribution to music in Chapter 12 of his fundamental work **Kitāb al-shifā',** also translated by d'Erlanger in *La Musique arabe,* Volume 2. A schematic interpretation of his method on the **'ūd** is given in Table 17.11.

Table 17.11 Ibn Sīnā-Avicenna (XIthc). Para-Pythagorean system on **'ūd**

A.	*To form a diatonic genre:*
A1	At a fourth of string, the little finger determines the fourth 150 mm; 4/3; 498°; 22 h; 10.K
A2	At a 1/9 of diapason from the nut, the forefinger defines the major tone 66.66 mm; 9/8; 203°9; 9 h; 4.E
A3	At a 1/9 of diapason of the forefinger-tailpiece, the ring finger indicates the ditone: (Pythagorean major third) 125.92 mm; 81/64; 407°8; 18 h; 8.1

A4 The rest between the ring finger and the little finger is a limma (baqīya): here:
150–125.92=24.08 mm; 256/243; 90°2; 4-h

B. To situate the "neutral" fingering degrees (low according to Avicenne)

B5 1/8 of diapason of the little finger-tailpiece (450:8=50, 25 mm) is related to the little finger's lower register (fourth minus a tone) for the middle finger called ancient or persian (Pythagorean minor third) here:
150–56.25=93.75 mm; 32/27; 294°1; 13 h; 5.F (forefinger+limma)

B6 Halfway between forefinger and little finger, some moderns attach a middle finger (low Zalzal's neutral third) 107.69 mm; 39/32; 343°; 15, 17 h; 7.H. The distance between the modern middle finger and the little finger is of 128/117, here 42.30 mm approximately. This middle finger is very low, nearly at string 40/33.

B7 At one major tone to this middle finger's low register, one places its "adjunct" (low neutral second) 46.15 mm; 13/12; 139°; 6.15 h; 3.D

B8 At this adjunct's low register, one places another adjunct: at one maxime tone (8/7) at modern middle finger's lower register (39/32), (high minor second) 37.36mm; 273/256; 111°5; 5 h; 2.C. It's called "head touch". It's a pseudo big harmonic semitone (37.5 mm; 16/15; 111°7; 5+h; 2.C) and a pseudo Pythagorean apotome (38.13 mm; 2187/2048; 113°7; 2.C). This "neutral" second, in fact high minor second, is nearly as low as the fingering degree 40/37 of the pre-Islamic system of the string division into forty aliquot sections. The "neutral" fingerings of Avicenna are therefore as low or nearly as low as their counterparts from the pre-Islam

Table 17.12

point 0:	nut. open string	
point 1:	from the nut, major second, major tone:	66.66 mm; 9/8; 203°9; 9 h
point 2:	from the nut, harmonic minor third: from point 1, big harmonic semitone:	100 mm; 6/5; 315°6; 14 h 48/45=16/15; 111°7; 4.94 h
point 3:	from the nut, p. major third, ditone: from point 1, major second, major tone: from point 2, middle harmonic semitone; pseudo-limma	125.92 mm; 81/64; 407°8; 18 h 9/8; 203°9; 9h 135/128; 92°2; 4+h
point 4:	from the nut, perfect fourth: from point 1, Pythagorean minor third: from point 2, harmonic minor tone: from point 3, Pythagorean limma:	150 mm; 4/3; 498°; 22 h 32/27; 294°1; 13 h 10/9; 182°4; 8 h+ 256/243; 90°2; 4-h
point 5:	from the nut, augmented 4a, Zarlino's tritone: from point 1, harmonic major third: from point 2, augmented harmonic second: also described by Fārābī on harps from point 3, harmonic minor tone: from point 4, small harmonic semitone; pseudo-limma	173.33 mm; 45/32; 590°2; 26.11 h 5/4=80/64; 386°3; 17 h 1350/1152=75/74; 274°6; 12.15h 10/9; 182°4; 8 h+

		135/128; 92°2; 4+h
point 6:	from the nut, Pythagorean perfect fifth:	200 mm; 3/2; 702°; 31 h
	from point 1, Pythagorean perfect fourth:	4/3; 498°; 22 h
	from point 2, harmonic major third:	5/4=80/64; 386°3; 17+h
	from point 3, Pythagorean minor third:	32/27; 294°1; 13 h
	from point 4, major second, major tone:	9/8; 203°9; 9 h
	from point 5, big harmonic semitone; pseudo-apotome	16/15; 111°7; 4.94 h

Paraharmonic system of al-Fārābī on the rabāba *(hurdy-gurdy) (tenth century)*

Al-Fārābī, whose theory applied to the *‘ūd* has been discussed, describes on the *rabāba* a complex paraharmonic system, of which we give in Table 17.12 the fingering degrees and the intervals which they define.[24]

Pythagoro-commatic system of al-Fārābī on the ṭunbūr *of Khurāsān (tenth century).*

The system studied by al-Fārābī on the *ṭunbūr* of Khurāsān is derived from the simple Pythagorean systems of Antiquity and the beginnings of Islam and it foreshadows the Pythagoro-commatic systems which were studied on the *‘ūd* from the thirteenth century by Ṣafī al-Dīn. In order to simplify the comparisons the strings are taken to have lengths of 600 mm.

In the simplest Pythagorean system, the elementary unit is the Pythagorean comma, the difference between twelve fifths and seven octaves, i.e. 8.07 mm, 531441/524288, 23°5, 1+h (see Table 17.13). Above is the limma, a rest interval between the ditone and the fourth, i.e. 30.47 mm, 256/243, 90°2. 4- h). Above again is the apotome, complementary to the limma to form a tone, and equal to a limma plus a comma, i.e. 38.13 mm, 2187/2048, 113°7, 5 h. The tone is therefore formed from a limma plus an apotome, i.e. two limmas and a comma, in the order limma-comma-limma or of course limma-limma-comma.

In the case of the scale of *ṭunbūr* of Khurāsān, the intervals will be in the form limma-limma-comma (the whole forming a tone) from the nut to the ninth (i.e. an octave and a tone). One finds five ‘normal’ frets (2a, 4a, 5a, 8a, 9a) and thirteen to twenty ‘variables’. Thus supposing that one is dealing with a scale of doh (C) one would have

Table 17.13 al-Fārābī (Xthc). Pythagorean-comma system on Khurāsān’s *ṭunbūr*

abbr 17/oc	*mm/600 reduced*	*numerical ratio*	*cents 1200/oct*	*Holder 53/oct*	*code JCC*	*name of the Pythagorean interval notes; hypothesis scale in C major*	
0	0				0.A	nut, open string	C
1.2am	30.47	256/243	90°2	4	1.B	limma; minor 2a	
2.2an	59.39	65536/59049	180°5	8	3.D	dilimma; diminished 3a; neutral 2a	

3.2aM	66.66	9/8	203°9	9	4.E	major 2a; major tone	D
4.3am	93.75	32/27	294°1	13	5.F	minor 3a	
5.3an	119.5	8192/6561	384°4	17	7.H	diminished 4a; neutral 3a	
6.3aM	125.92	81/64	407°8	18	8.1	major 3a; ditone	E
7.4aj	150	4/3	498°0	22	10.K	perfect 4a	F
8.5ad	172.85	1024/729	588°3	26	11.L	diminished 5a	
9.4aA	178.6	729/512	611°7	27	12.M	augmented 4a; tritone	
10.5aj	200	3/2	702°0	31	14.0	perfect 5a	G
11.6am	220.3	128/81	792°0	35	15.P	minor 6a	
12.5aA	225.4	6561/4096	816°0	36	16.Q	augmented 5a	
13.6aM	249.1	14348907/8388608	930°0	40	18.S	major 6a	A
14.7am	262.5	16/9	996°0	44	19.T	minor 7a	
15.6aA	267	59049/32768	1019°6	45	20.U	augmented 6a	
16.7aM	284	243/128	1109°8	49	22.X	major 7a	B
17.8aj	300	2/1	1200°0	53	24.Z	perfect 8a; octave	C
18.8a+	304	531441/524288	1223°5	54	1.B	Pythagorean comma (+8a)	
19.8aA	319	2187/2048	1313°7	58	2.C	Pythagorean apotome (+8a)	
20.9aM	333.33	9/8	1403°9	62	4.E	major 2a; major tone (+8a)	D

do♮-limma-réϬ-limma-réd-comma-*ré*♮-limma-miϬ-limma-mid-comma-mi♮
mi♮-limma-*fa*♮-limma-solϬ-comma-fa∓-limma-*sol*♮-
sol♮-limma⊀laϬ⊁comma-sol∓-limma-la♮-limma-siϬ-comma⊀la∓⊁
limma-si♮
si♮-limma-*do*♮
do♮-comma-do+-limma-do∓-limma-*ré*♮

Note that the signs laϬ and la∓ are not used.

There are therefore seventeen fingering degrees per octave and seventeen intervals per octave, formed from commas, limmas apotomes, dilimmas or major tones etc. The instrument has a range of major ninth on the string. Between the two strings, the tuning recommended by al-Fārābī envisages the unison (chord ‘bride’) the limma, the dilimma of the ‘mountain people’, the tone, the minor third of Bukhārā, the fourth as the *ʿūd* or even the fifth. The neutral degrees are very raised, much more than in the descriptions of Zalzal, al-Fārābī or Avicenna on the *ʿūd*.[25]

Pythagoro-commatic system of Ṣafī al-Dīn on the ʿūd[26] *(thirteenth century)*

Ṣafī al-Dīn al-Urmawī al-Baghdādī (born near Urmīya, raised in Baghdad, died in 1284), protégé of the last Abbasid Caliph and then, after the fall of Baghdad in 1258, spared and protected by the Mongol conquerors, led the Pythagorean system based on the cycle of fifths at its peak.

In the *Kitāb al-adwār (Book of Cycles)* and in the *Risāla al-Sharafīya (Epistle to Sharif),* he proposes a Pythagoro-commatic solution called ‘systematic’ with a view to inserting the neutral, native and empirical fingering degrees onto the scale of sounds. He accepts, as does al-Fārābī, on the *ṭunbūr* of Khurāsān, the division of the major tone into two limmas and a comma, that of the fourth into two tones and a limma, and that of the octave into two fourths and a major tone. It is therefore exclusively Pythagorean (Table 17.14).

It is necessary to note the originality of the system by Ṣafī al-Dīn, who, in strictly respecting the Pythagorean calculations derived from the cycle of fifths, succeeds in assigning the empirical intervals to the origin.

Thus the neutral second has become a diminished third, dilimma 59.39 mm, 65536/59049, 180°5, 8 h, 3.D, not to be confused with the very near minor harmonic tone (60 mm, 10/9, 182°4, 8 h, 3.D) or with the indicator (60mm, 40/36, 182°4, 8 h, 3.D) of the division into forty, even though the practitioners of music could confuse these two derived intervals of the three different systems (see Table 17.15).

Table 17.14 Pythagorean-comma system from the XIIIthcentury. Ṣafī al-Dīn

code JCC	*ratio*	*cents Farmer*	*comm Hold*	*mm /600 string*	*fingering degree*	*commentary, equivalence (cf comparative table register "7")*
1.B	256/243	90° 2	4	30.47	surplus (forefinger)	limma (pre-Islam)
3.D	65536/59049	180°5	8	59.39	neighbour-forefinger	Pythagorean minor tone; “dilimma”
4.E	9/8	203°9	9	66.66	forefinger	major tone (Pythagoras)
5.F	32/27	294°1	13	93.75	Persian middle finger	minor 3a (Pythagoras)
7.H	8192/6561	384°4	17	119.46	Zalzal middle finger	diminished Pyth. 4a; ps. harmonic major 3a

8.1	81/64	407°8	18	125.92	ring finger	major 3a (Pythagoras)
10.K	4/3	498°0	22	150	little finger	perfect fourth
11.L	1024/729	588°3	26	172.85		5a diminished, Pythagoras
13.N	262144/177147	678°5	30	194.5		fifth –1 comma (pre-Islam)
14.0	3/2	702°0	31	200		perfect fifth

The neutral third has become a diminished fourth, 119.45 mm, 8192/6561, 398°4, 17 h, 7.H, not to be confused with the very near major harmonic third (120 mm, 5/4, 386°3, 17 h, 7.H) or with the indicator (120 mm, 40/32, 386°3, 17 h, 7.H) of the division into forty, even though the practitioners of music could confuse them.

On this subject, if a great Western musicologist said about Ṣafī al-Dīn that he was the Zarlino of the East,[27] this laudistic comparison is erroneous because Ṣafī al-Dīn is on the contrary the one who attracted the best applications of the potentials of the Pythagorean system by putting the placement of the neutral fingerings on an inversion of the diatonic genre starting from the Pythagorean diminished 5a:172.85 mm, 1024/729, 588°3, 26 h, 11. And he comes to situate them at the level of two indicators of the old system of division of the string into forty aliquot segments from which these two fingering positions are perhaps derived.

This system, highly appreciated, has been carried on by the contemporaries and the successors of Ṣafī al-Dīn such as al-Shīrāzī (thirteenth century, al-Jurjānī and Ṣafī al-Dīn (fourteenth century).[28] It could be asked, after many mentions of scholarly treatises, what the real impact of these treatises was on the musical practices of the Arabo-Islamic world at the time of their conception. What were the popular musicians doing? Were they able to understand the paraharmonic system of al-Fārābī on the *rabāba,* the Pythagoro-commatic system of Ṣafī al-Dīn on the *ʿūd*? Or were they staying with the empirical longitudinal divisions according to Zalzal's method dating from the eighth century?

Table 17.15

LUTH-ʿŪD. *INTERPRETATION OF THE METHOD BY ṢAFĪ AL-DĪN TO LOCATE SOUNDS*

A. FORMATION OF THE DIATONIC GENRE FROM LOW TO HIGH REGISTER

1. One substracts 1/9 of the string from the nut/A and the forefinger, in D, marks the first major tone: 66, 66 mm; 9/8; 203°9; 9 h; 4.E.
2. One substracts 1/9 from the remainder Forefinger/tone/D—tailpiece/M; and the ring finger, in Z, marks the major 3a or ditone: 125, 92 mm; 81/64; 407°8; 18 h; 8.1.
3. The distance between ditone/Z and perfect fourth (150 mm; 4/3; 498°; 22 h; 10.K) H', defines the remainder or limma: here 24, 08mm; 256/243; 90°2; 4 h.

B. FORMATION OF THE REVERSE DIATONIC GENRE FROM HIGH TO LOW REGISTER

4. One calculates from the remainder of the string fourth/ring finger/H'—tailpiece/A (thus 3/4 of the string 450 mm), 1/8 of that remainder (thus 56, 25 mm) and one carries forward that value from the fourth to the nut/A. Here, the 'ancient middle finger' marks the minor 3a: 93, 75 mm; 32/27; 294° 1; 13 h; 5.F. One lowers it down by a major tone.
5. One calculates from the remainder of the string minor 3a/H—tailpiece/A (thus 506, 25 mm), 1/8 of that remainder (thus 63, 28 mm) and one carries forward that value (a tone) from the minor 3a/middle finger/H to the nut/A. Here, in B, the 'za'id-surplus' neighbouring the forefinger marks the limma/minor 2a:30, 47mm; 256/243; 90°2; 4 h; 1.B.
6. It is a limma as one has substracted a ditone from the fourth, in order to obtain that remainder A–B. (two spaces from a tone)

C. PYTHAGOREAN DETERMINATION OF NEUTRAL FINGERING DEGREES

7. One calculates from the remainder of the string limma/B—tailpiece/A (thus 569, 53 mm), 1/4 of that remainder, hence a perfect fourth (thus 142, 38 mm), and one transposes that value to the nut/M. Here, in T', one obtains a limma plus a fourth, thus a diminished 5a: 172, 85 mm; 1024/729; 588°3; 26 h; 11.L.
8. One calculates from the remainder of the string diminished 5a/T'—tailpiece/A, (thus 427, 15 mm), 1/8 of that remainder (thus 53, 39 mm) and one carries forward that value from the diminished 5a/T' to the nut/A. Hence one lowers it by a tone (9/8) and at the defined place, W, the middle finger of Zalzal marks the diminished 4a or neutral 3a: 119, 45 mm; 8192/6561; 384°4; 17 h; 7.H.; major third minus a comma.
9. One calculates from the remainder of the string diminished 4a/W—tailpiece/A (thus 480, 55 mm), 1/8 of that remainder, (thus 60, 06 mm) and one carries forward that value from the diminished 4a/W to the nut/A. Hence one lowers it by another tone (9/8) and at the defined place J, the neighbour of the forefinger marks the diminished 3a or neutral 2a, dilimma: 59, 39 mm; 65536/59049; 180°5; 8 h; 3.D.; major second minus a comma.
10. Empirical neighbour of the forefinger half-way limma—major 2a=48, 56 mm.
11. Empirical neighbour of the forefinger half-way nut—minor 3a=46, 87 mm.
12. Empirical neighbour of the forefinger half-way nut—neutral 3a=59, 72 mm.
13. Empirical neutral middle finger half-way major 2a—4a=107, 69 mm; 39/32.

CONCLUSION

With the quasi-perfection of the Pythagoro-commatic system conceived by al-Fārābī on the *ṭunbūr* of Khurāsān in the tenth century, by Ṣafī al-Dīn on the *ʿūd* in the thirteenth century, and carried on by al-Jurjānī (fourteenth century), Ibn Ghaybī Maraqī and Shukrullāh (fifteenth century), al-Lādhiqī (sixteenth century), it can be lamented that this system started to regress in the fifteenth century and practically disappeared in the Arabo-Persian world and has only been carried on in Turkey.

From the thirteenth century, the Arabo-Persian world was subjected to the impact of encountering a much stronger Western world in the guise of a compromise of musical writing represented by the European staves and the inflections in fourths of a tone. It is henceforth the way that Arabic music is taught and diffused. But all hope is not lost. The twentieth century has been marked by the Meeting in Cairo in 1932 of the First Congress of Arabic Music. Reference to the language of the *ʿūd* has been practically lost, but it is known how to make an inventory of the styles and rhythms. The art and science of music are taught and preserved. That is essential.

NOTES

1 Basic work of Rodolphe d'Erlanger (1930:I, 218–42, *tunbūr* of Baghdad).
2 *ibid.*
3 Cf. Farmer, article **'mūsīḳī'**, *Encyclopédie de l'Islam,* 1st edition (1960), p. 801 and Mehdi Barkechli.
4 Cf. Farmer, article **'mūsīḳī'**, *Encyclopédie de l'Islam,* p. 801.
5 Farmer, who calls to mind three different manuscripts of al-Kindī, brings out the influences of Euclid and Ptolemy on the last of these three manuscripts. The interpretations of the theories of al-Kindī are not all concordant, even under the pen of Farmer. In the article **'mūsīḳī'**, *Encyclopédie de l'Islam,* and in 'Arabian Music', *Groves Dictionary of Music and Musicians,* Farmer gives two different explanations of al-Kindī's theory.
6 With regard to this system of Pythagoras in Islam (**al-Mawṣilī,** al-Munajjim, al-Kindī, **al-Iṣfahānī**) consult Rouanet, J. 'La musique arabe', in Lavignac and de la Laurencie (eds), *Encyclopédie de la Musique et Dictionnaire du Conservatoire,* Paris, 1922, I, 5, pp. 2701–4, [etc. list of books and authors…]. We note that certain Arab authors contemporary and nationalist seem grieved that this system is Pythagorean and would like to find it in Semitic or Arabic origins.
7 Cf. **al-Iṣfahānī** V, p. 270 or V, p. 53 (ed. Būlāq: cited in d'Erlanger (1939) p. 377).
8 One finds mention of Pythagorean tritone in al-Fārābī, *Kitāb al-mūsīqī,* translation by d'Erlanger (1930) Book II. Instruments, harps pp. 286–304.
9 d'Erlanger (1930, 1935:1–101).
10 d'Erlanger (1930: Introduction, pp. 1–77).
11 *Ibid.,* Book 1, pp. 79–114.
12 *Ibid.,* Book 1, pp. 115–62.
13 *Ibid.,* **ʿūd,** Essay 1, pp. 163 *et seq*.
14 *Ibid.,* p. 171.
15 *Ibid.,* p. 179.
16 *Ibid.,* p. 204. The last described fingering degree, a limma under the index, is the Pythagorean apotome 2187/2048. It is seen therefore that the practice of the shift, already recommended by **Isḥāq al-Mawṣilī** well before al-Fārābī, formed part of the most ancient techniques of Arabic music. In these conditions, the slide used by the lutenists of the School of Baghdad in the twentieth century will not have to incur the reprobation of the Arabs who call themselves traditionalists.

17 *Ibid.,* Book 2, Essay 2, pp. 218 *et seq*.
18 *Ibid.,* p. 242 *et seq*.
19 *Ibid.,* p. 262 *et seq*.
20 The name *tritone of Zarlino,* designed to facilitate understanding, can scarcely be applied to an interval played in the tenth century.
21 *Ibid.,* p. 277 *et seq*.
22 *Ibid.,* p. 286 *et seq*.
23 For remarks on the theories on calculation of Ibn Sīnā (Avicenna), cf. H.G. Farmer, article **'mūsīḳī',** *Encyclopédie de l'Islam,* 1st ed. pp. 801–807 and (1937); d'Erlanger (1935), pp. 234–237, diagram p. 236; J.C.Chabrier (1976), Book 2, pp. 382–3.
24 The indicators 4 (perfect fourth) and 6 (perfect fifth) are declared to be optional. Perhaps they stem from the logic of al-Fārābī, who has already entered into the system of the division of the string into forty aliquot segments in order to rationalize them. But here, the complexity of the system is very great and one could ask oneself how the player of the *rabāba* of the tenth century had the notions of an acoustic nature to apply such a system. The system is perhaps entirely reliant on a tributary of the imagination.
25 For remarks on the ***tunbūr*** of Khurāsān, consult the translation by d'Erlanger (al-Fārābī) t. 1, pp. 242–62. This system of seventeen indicators foreshadows that of the long-necked lutes of the twentieth century. Cf. record 'arabesquesrecitalbum no 8, luth traditionnel au Liban. (Arab recital album no 8, traditional lute of the Lebanon) Buzuq. Nasser Makhoul, see diagram/outline JCC.
26 For remarks on the theories and methods of calculation of **Ṣafī al-Dīn** on the ***'ūd,*** cf. Farmer, **'mūsīḳī',** *Encyclopédie de l'Islam,* and d'Erlanger (1938), preface V-VI, VIII-IX; **Ṣafī al-Dīn,** *Risāla al-Sharifīya* Epître a Sharif—Epistle to Sharif, translated by d'Erlanger (1938) ***'ūd,*** pp. 111 *et seq*. (essential calculation). Comments on the *Kitāb al-adwār*—Book of cycles of **Ṣafī al-Dīn,** translated by d'Erlanger (1938), pp. 481 (tunings), 308, 580, 603 (transpositions), 411 (courses of strings), 551 (plectrum technique), 595 (nuances).
27 Kiesewetter, cited by Farmer, **'mūsīḳī',** *Encyclopédie de l'Islam,* p. 804.
28 Al-Jurjānī, *Commentaire sur le Kitāb al-adwār* (Commentary on the Kitāb al-adwār), translated by d'Erlanger (1938) pp. 220 *(et seq.)*. Anonymous, *Traité anonyme dédié au Sultan* (Anonymous treatise dedicated to the Sultan), translated by d'Erlanger (1939), pp. 27 (et seq.) al-Lādhiqī, ***Risāla al-Fatḥīya,*** translated by d'Erlanger (1939), pp. 291 *(et seq.)*.

18
Statics

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Statics as a separate scientific discipline—'the science of weighing'—was formed in Antiquity.

Its main problem was initially the calculation of the gain in force applied by the use of special mechanical devices. The Greek word μηχανή *(mechane)* meant initially a machine as a variety of ingenious devices. Accordingly, the term 'mechanics' related to the science of 'simple machines', i.e. those intended for moving large loads by applying a small force. The Greeks ranked statics as high as arithmetic, 'the science of counting', and subdivided both into the theoretical and the practical (applied).

In the times of Antiquity, two trends appeared in statics: the geometrical, which was of a theoretical nature, and the kinematic, of a practical nature (Duhem 1905/6: vol. I, p. 16). In the former case the laws of equilibrium were studied using a stationary balanced lever. The concept of centre of gravity was also introduced in association with the geometrical trend in statics, which was characterized by a high degree of mathematization of the theory.

The basis of the kinematical trend in statics was the practical application of 'simple machines' for the lifting and moving of loads. In that case, the laws of equilibrium of bodies were studied using an unbalanced lever. The conclusions from the principal theorems of statics were based on certain explicitly or inexplicitly adopted assumptions from the field of dynamics. This trend ascends to the *Mechanical Problems* of pseudo-Aristotle *(Mechanica)*.

The Greeks divided all kinds of mechanical motion into 'natural', i.e. occurring *per se,* without any external action (such as the falling of heavy bodies), and 'forced' or violent, i.e. those occurring under such actions (δύναμις). The stimulus for 'natural motion' was thought as a certain 'tendency' or 'inclination' (ῥοπή) inherent in a body proper. The initial problems of the Greek statics were, firstly, to determine this 'inclination' (ῥοπή) and, secondly, to find the centre of gravity of bodies. Both problems were stated and solved by Archimedes, who gave a strict mathematical formulation of the law of lever and defined the centre of gravity as a point of a body characterized by the body remaining in equilibrium when fastened at that point. For this reason alone, Archimedes should be regarded as the real founder of statics as a theoretical discipline.

Archimedes determined the centre of gravity of a single body and of systems of two and three bodies and then proved the principal law of lever which was formulated as follows: 'commensurable and incommensurable quantities are balanced at lengths inversely proportional to their weights'.

The origin of hydrostatics also ascends to the era of Antiquity. Here, again, Archimedes was the first to propose the theory of equilibrium of bodies in liquids and to study the stability of that equilibrium.

The formation of the kinematic trend in statics dates to the late Hellinistic period. A lever in this case was studied at a moment when its equilibrium was disturbed.

Thus, the essence of the two trends in the antique statics can be briefly formulated as follows: in the former case, the methods of Greek geometry were applied to a stationary balanced lever; in the latter case, the motion of the ends of an unbalanced lever was reduced to the motion of a point over a circle.

THE PREHISTORY OF ARABIC STATICS

In the history of medieval mechanics, statics was probably the discipline that was influenced most strongly by the antique tradition. In statics, it is possible to trace even chronologically the process of assimilation of the scientific heritage of Antiquity.

The beginning of the geometrical and kinematic trends in medieval statics ascends to the comments and treatments of the works of Archimedes, Aristotle, Hero of Alexandria, Pappus of Alexandria and Vitruvius. Translations and comments of Aristotle's works were of substantial importance.

It is still unknown whether Archimedes' treatises on mechanics and pseudo-Aristotle's *Mechanical Problems* were translated into Arabic. In any case, such translations still have not been discovered. On the other hand, a number of anonymous treatises of the late Alexandrian period, translated into Arabic or from Arabic into Latin, have survived up to our time (some of them are ascribed to Euclid and Archimedes). It should be noted that these treatises were translated in the first place in medieval Europe at the point when the period of assimilation of the antique and oriental scientific heritage commenced there. Along with the Syriac and earlier Arabic translations of the antique author these treatises were an object of keen interest and study in the medieval East and later in Western Europe. They were essentially and chronologically an 'intermediate link' between the mechanics of Antiquity and that of the Medieval East. Among them, three anonymous treatises of Greek origin, which are known to exist in Arabic translations, deserve special attention:

1 The treatise attributed to Euclid and titled *Maqāla li-Uqlīdis fī al-athqāl (The Book of Euclid on the Balance)* (Woepcke 1851:217–32; Clagett 1959:24–30).
2 The treatise *Liber de canonio (The Book Concerning the Balance)* translated from the Greek directly into Latin and devoted to the unequal-arm (Roman) balance (Moody and Clagett 1952:55–76).
3 The anonymous treatise *Liber Euclidis de ponderoso et levi et comparatione corporum ad invicem (The Book of Euclid on the Heaviness and Lightness)* which has survived to our time in the Arabic and Latin versions (Moody and Clagett 1952:23–31).

In addition, there was a treatise in Arabic translation, *Maqāla li-Arshimīdis fī al-thiql wa-l-khiffa (On Heaviness and Lightness)* (Zotenberg 1879; Moody and Clagett 1952:52–5), a brief account of the first part and the first proposition of the second part of Archimedes' treatise *On the Floating Bodies*. It contains only the formulations of Archimedes' proposals (without proofs).

In contrast to the *Mechanical Problems,* Hero's treatises and other works of the Alexandrian period where the principal law of lever was proved in an explicit or inexplicit kinematic approach, *The Book of Euclid on the Balance* was written along the traditions of Archimedian geometrical statics.

The formulations and proofs are sometimes very close to the methods employed in Euclid's *Elements,* but the book is undoubtedly even closer to Archimedes' style and methods and, more strictly, to his work *Equilibrium of Planes*. Unlike Archimedes, the anonymous author passes on from the planar to the three-dimensional problem: the lever is regarded as a real homogeneous beam, rather than as a geometric line. The principal law of lever, however, is proved only for commensurable loads.

The second treatise, *Liber de canonio,* written somewhat later than pseudo-Euclid's treatise but closely associated with it, is the next step in his history of geometrical statics. Proceeding from the law of lever for a weightless beam with commensurable loads, the author of *Liber de canonio* passes on to analysing the equilibrium conditions for a homogeneous ponderable beam with a load suspended from its shorter arm. Thus, the essence of this treatise consists in developing the principal idea of pseudo-Euclid which was to consider the weight of the beam. The proof, however, which the author of *Liber de canonio* based on the assumption that the weight of a portion of a lever beam having a uniform thickness and made of a homogeneous material was equivalent to the weight of a load applied in its middle, was the result of application of Archimedes' theory of the centre of gravity to a real ponderable lever.

Thus, *Liber de canonio* is close to pseudo-Euclid's treatise and is successive to this in its contents. Further, it is close to one of the classical Arabic treatises, *Kitāb al-qarasṭūn* (*Liber karastonis* in the Latin version) of Thābit ibn Qurra, which it precedes chronologically. This makes it possible to relate *Liber de canonio* to the initial stage of the development of statics in the medieval East.

Unlike Archimedes, however, who reduced real bodies to geometrical abstractions (lines and plane figures), the authors of these treatises start to apply Archimedes' classical theory of the weightless lever to real problems on equilibrium and weighing, though the method of presentation and the principles of proving remain Archimedean in their essence, as well as in their form.

The Book of Euclid on the Heaviness and Lightness is actually a treatment of a few of Aristotle's books. It explicates Aristotelian concepts of place *(locus),* magnitude *(magnitudo),* kind *(gens)* and force *(virtus fortitudo)*.

This treatise served more than the original of Aristotle's works as the basis for interpreting the concepts of force and weight, and also as the basis of the theory of motion in 'filled' media which was later developed in the medieval East.

The treatise *On Heaviness and Lightness,* together with the introductory part of Menelaus' treatise on determining the composition of alloys by means of hydrostatic balance, laid down the foundations of hydrostatics at that period.

Another trend in the late-Alexandrian statics was represented by traditional practical guides and instructions for constructing mechanical devices. It ascends to *Mechanical Problems* and the works of Philon, Hero of Alexandria, Vitruvius etc. and led to practical (applied) statics. These included both translations of antique authors and the earlier treatment of their works (for instance, Hero's *Mechanics* was translated into Arabic by Qusṭā ibn Lūqā al-Baʿalbakkī in the ninth century).

PRINCIPAL TRENDS IN ARABIC STATICS

The sources

Three principal trends can be distinguished in Arabic statics:

1 theoretical statics as a tradition of Archimedes and *Mechanical Problems* in combination with Aristotle's 'dynamic principle' and the associated science of weights;
2 hydrostatics and the science of specific weights; and
3 the science of ingenious devices (*ʿilm al-ḥiyal,* a literal translation of the Greek μηχανή), which apart from the science of constructing 'simple machines' and their combinations, also includes the 'science of water lifting'. Exactly this narrower sense was given to 'mechanics' in the majority of medieval oriental encyclopedias.

We now know of more than sixty works on statics written in the medieval East in Arabic or Persian. Among them there are works whose authorship is undoubtful, others are anonymous, whereas some works and treatises have survived up to our time only because they turned out to be included in the works of other authors.

The majority of these works are in the field of 'practical statics' *(ʿilm al- ḥiyal)*. These include Kitāb *al-ḥiyal (The Book of Ingenious Devices)* of the Banū Mūsā (ninth century), which engendered a large number of commentaries and treatments; *Kitāb fī maʿrifat al-ḥiyal al-handasiyya (The Book of Knowledge of Ingenious Mechanical Devices)* of al-Jazarī (twelfth century); *Miʿyār al-ʿaql (The Measure of Mind)* of Ibn Sīnā (eleventh century); and chapters on mechanics in Ibn Sīnā's encyclopaedic works, which were based on *Mechanical Problems* and Hero's *Mechanics*. A section on mechanics was traditionally included in the contents of the majority of medieval scientific encyclopaedias. The most comprehensive among them was *Mafātīḥ al- ʿulūm (The Keys of Sciences)* by ʿAbd Allāh al-Khwārizmī, with one chapter fully devoted to mechanics. In some of the encyclopaedias, the 'science of water lifting' was described under a separate entry and regarded as a division of geometry.

The works of a theoretical nature were less numerous. Those worth mentioning include, firstly, a cycle of treatises on *qarasṭūn* (the non-equal-arm balance), with Kitāb *al-qarasṭūn* by Thābit ibn Qurra (ninth century) being the most significant among them historically and scientifically, and secondly, *Kitāb mīzān al-ḥikma (The Book of the Balance of Wisdom)* by al-Khāzinī (twelfth century), which is by right considered an encyclopaedia of statics of the medieval East. The author included in the book many abstracts from the works of his predecessors, which have been lost for us, such as al-Qūhī (tenth century), Ibn al-Haytham (tenth-eleventh century), al-Bīrūnī (tenth-eleventh century), al-Rāzī (eleventh century), ʿUmar al-Khayyām (eleventh-twelfth century) etc.

A third, rather large group of treatises is devoted to the problem of determining the specific weight of metals and minerals, including theoretical and practical solutions of the problem. These questions are the central ones in al-Khāzinī's treatise and were dealt with in special works of al-Bīrūnī *(Maqāla fī nisab)*[1], al-Nayrīzī (see Wiedemman 1970a) and ʿUmar al-Khayyām (*Kitāb mīzān al-ḥikma,* pp. 87–92) and in the works of their predecessors and followers.

THEORETICAL STATICS

As follows from the foregoing, the principal problems of statics dealt with in the Medieval East were its axiomatics, the concepts of force, weight and gravity, the theories of lever and the centre of gravity, equilibrium and its stability and hydrostatics.

It should be noted, however, that the problems of theoretical statics can hardly be distinguished from the problems of dynamics of that period, and not only because the science of statics was based on the synthesis of the geometrical and dynamic traditions of the antique mechanics, but also since the scientists of the medieval East generalized some principles of statics and applied them to moving bodies. In that case, the antique teaching on motion, which belonged fully to the philosophical tradition, was mathematized and constructed in the spirit of Archimedes' geometrical statics. For that reason, some concepts of mechanics, such as force, weight, centre of gravity, centre of the universe etc., should be considered in two aspects: statical and dynamical.

Weight and gravity, force

The concepts of force and weight are treated in the mechanics of the medieval East in three ways:

1 in the Aristotelian sense, in connection with the concept of 'natural place' and 'centre of the Universe';
2 in the Archimedean sense, in connection with the principal concepts of geometrical statics; and
3 as applied to the Aristotelian theory of motion of bodies in a 'filled medium'.

The third aspect will not be considered here, since it relates to motion of bodies, rather than to their equilibrium. Thus, there remain two aspects of the concepts of force and gravity, which will be discussed below. There are two sources which can be used to monitor the progress of these concepts in Arabic mechanics: *Kitāb* ***al-qarasṭūn*** by Thābit ibn Qurra and *Kitāb mīzān* ***al-ḥikma*** by al-Khāzinī. The latter contains abstracts from the works of antique authors and al-Qūhī (tenth century), Ibn al-Haytham (tenth-eleventh century) and al-Isfizārī (eleventh century) on theoretical statics, in addition to the author's own results.

To begin with, the authors cited differentiate between the weight *(wazn)* of a body and its 'gravity' *(thiql)*. The weight of a body is constant and can be measured by weighing it. Following the antique traditions, these authors associated the weight of a body with the pressure exerted by a load on the scale during weighing. As to 'heaviness', they considered it a certain variable category. It could vary depending on the position of a body relative to a particular point which might be either 'the centre of the universe' or the axis of rotation of a lever.

If the 'gravity' of a body is associated with the body's position relative to the 'centre of the universe', this notion ascends to the Aristotelian concepts of 'natural motion' and 'natural place.'[1]

If, however, this concept is associated with the position of the load on the lever arm, it is connected with the idea expressed by the author of *Mechanical Problems* that one and the same load 'draws down' differently depending on its position on the lever arm.

Further, the authors cited associate the concept of 'gravity' with the concept of 'force' *(quwwa)*. This correlation is established as follows:

> A heavy body, [states al-Khāzinī, following al-Qūhī and Ibn al-Haytham] is a body which moves towards the centre of the Universe under the action of the force contained in it. This force moves the body only to the centre of the Universe and in no other direction. It (the force) is inherent in the body and does not leave the body until this reaches the centre of the Universe.
>
> (al-Khāzinī, *Kitāb mīzān al-ḥikma*, English translation, p. 16)

This definition is purely Aristotelian. The point is that the 'body' is in a 'natural motion' towards its 'natural place', the centre of the universe. The force is understood as a 'tendency' or inclination (Arabic *mayl*), i.e. a certain capability of the body for performing action; in this sense the term is analogous to the Greek ῥοπή. Al-Khāzinī further establishes the relationship between this 'force' and the physical properties of a heavy body: the density, volume and shape.

1 Heavy bodies may possess different force. Bodies of a greater density possess a higher force.
2 Other bodies possess a lower force. These are bodies of lower density.
3 The higher is the density, the greater is the force.
4 Bodies equal in the force possess the same density.
5 Bodies equal in the volume, identical in the shape, and equal in their heaviness, are equal in their force.

(*Kitāb mīzān al-ḥikma*, English translation, p. 16)

These five propositions in al-Khāzinī's treatise are identical to the seventh-ninth axioms in the above-cited pseudo-Euclid's treatise *The Book of Euclid on the Heaviness and Lightness* which was fully included in *Kitāb mīzān al-ḥikma*. It is most likely that this book together with Aristotle's *Physics* was among the principal works on which al-Qūhī and Ibn al-Haytham based themselves.

Since the gravity of a body is associated with the 'force' and the latter leaves the body as it reaches the 'centre of the universe', the 'gravity' should be zero at that centre. Thus, 'gravity' was thought to be a certain variable category. As to the distance of a body from the 'centre of the universe', it was defined as a straight line section connecting the centre of gravity of a body with the 'centre of the universe'.

Al-Qūhī and Ibn al-Haytham indicated that the 'gravity' of a body certainly depended on that distance. Bodies of the same heaviness were defined as those equal in the 'force', volume, shape and distance from the 'centre of the universe'. On the contrary, if bodies had the same 'force', volume and shape, but were at different distances from the 'centre of universe', they possessed different 'gravities'.

'Two heavy bodies equal in the force, volume and shape, but positioned at different distances from the centre of the Universe, possess different heaviness. The more distant body is heavier' (Kitāb mīzān al-ḥikma, English translation, p. 20). As may be seen, al-Qūhī and Ibn al-Haytham proceed here from *The Book of Euclid on the Heaviness and Lightness*.

This proposition is further developed by al-Khāzinī himself. He states as follows:

> For each heavy body of a known weight positioned at a certain distance from the centre of the Universe, its gravity depends on the remoteness from the centre of the Universe. The farther is a body from the centre of the Universe, the heavier it is; the closer it is to the centre, the lighter it is. For that reason, the gravities of bodies relate as their distances from the centre of the Universe.
>
> (*Kitāb mīzān* al-ḥikma, English translation, p. 24)

According to al-Khāzinī, this variation of the gravity of a body with its distance from the 'centre of the universe' is associated with variations of density of the 'cosmos', i.e. the medium surrounding the Earth from the maximum at the Earth's surface to zero at the 'periphery' of the cosmos and vice versa. The 'gravity' of a body is understood here as a category similar to the modern concept of potential energy (Rozhanskaya 1976:146).

Thus, the author of *Kitāb mīzān* al-ḥikma was the first in the history of mechanics to propose the hypothesis that the gravities of bodies vary depending on their distances from the centre of Earth. Neither of the medieval treatises known at present considered this problem.

Another aspect of the concept of 'gravity' is associated with the use of the same term *(thiql)* to denote a load suspended from an end of a lever. In this case too, we should refer first of all to Thābit ibn Qurra's Kitāb al- qarasṭūn where he suggests two formulations of the law of lever. The first formulation ascends to the statement expressed by the author of *Mechanical Problems,* that one and the same load possesses different 'gravity' depending on its position on the arm of a lever. As regards the second formulation, Ibn Qurra employs the strict methods of the antique mathematics to consider successively the equilibrium of two loads on a weightless lever, the equilibrium of an arbitrary number of loads and, finally, the equilibrium of a continuous load, and arrives at the definition of the centre of gravity of a heavy section. In both cases, the gravity of a body is correlated with the body's position on the lever. According to Ibn Qurra, heaviness can vary depending on that position. For instance, a load on a longer arm of a lever exerts a higher pressure (i.e. has a greater 'gravity') than the same load on the shorter arm. In this case, 'gravity' is understood essentially as the moment of a force relative to a point.

Al-Qūhī, Ibn al-Haytham and, after them, al-Khāzinī combine the two aspects of the concept of 'gravity': that associated with the 'natural tendency' and the distance of a body from the 'centre of the universe' and that connected with the distance of a load from the point of support or suspensions of a lever.

In both cases, the weight, or gravity, of a body depends on the position of that body relative to a particular point.

The first aspect was not developed in medieval mechanics in the East or in the West. The phenomenon of variation of the gravity of bodies with variations of their distances from the centre of the Earth was discovered only in the eighteenth century after a certain development in the theory of gravitation.

The second aspect can be considered as the prototype of the later concept of *gravitas secundum situm* (positional gravity). The concept of 'positional gravity' was widely used in medieval European statics, in particular in the works of Jordanus de Nemore and his pupils and followers (Moody and Clagett 1952:69–112, 182–90; Rozhanskaya 1976:147).

It was Jordanus de Nemore who postulated the difference between weight as a constant and gravity as a variable category which was characteristic of the statics of the medieval East.

Note in conclusion that the Latin terms *pondus* (weight) and *gravitas* (heaviness) are most probably literal translations of the Arabic words *wazn* and *thiql*.

The centre of gravity

As has already been mentioned, the concept of the centre of gravity appeared first in Archimedes' works. According to Archimedes, the centre of gravity of a body is a particular point within that body such that, if the body is fixed (suspended) at that point, it will be at rest and retain the initial position, since all planes drawn through that point divide the body into mutually balanced portions.

Archimedes elaborated methods for determining the centre of gravity of a single body and of a system of bodies, but reduced the problem to a purely geometrical one by replacing a real body or system of bodies by plane figures.

In the treatises of al-Qūhī, Ibn al-Haytham and al-Isfīzārī, the classical results of Archimedes are applied to three-dimensional bodies and systems of such bodies. The authors mentioned set forth almost all of Archimedes' axioms relating to the centre of gravity, but as applied to real ponderable bodies.

Al-Qūhī and Ibn al-Haytham state the following:

1 If two heavy bodies are connected together in such a manner that their positions relative to each other remain unchanged, the combination of bodies will have a common centre of gravity and the point will be unique.
2 If two heavy bodies are connected together by means of a third body whose centre to gravity lies on the straight line connecting the centres of gravity of the two bodies, the centre of gravity of the whole combination of bodies will lie on the same line.
3 If a heavy body counter-balances another heavy body, then any third body of a heaviness equal to that of the second body will counter-balance the first heaviness, provided that the positions of the centres of gravity of each of the bodies is not changed.
4 Let two bodies be balanced. If one of them is removed and a heavier body is placed into the centre of gravity, the latter will not counter-balance the remaining body. It will counter-balance a heavier body than the remaining one.
5 For two heavy bodies connected with each other, the ratio of their heavinesses is an inverse of the distances of their centres of gravity from the common centre of gravity of their combination.

(al-Khāzinī, *Kitāb mīzān al-ḥikma*, English translation, pp. 19–20)

This system of axioms is then supplemented with three statements which are valid only for three-dimensional figures: a rectangular prism and a parallelepiped (bodies 'with parallel faces and similar portions').

1 The centre of gravity of a body with parallel faces and similar portions is its [geometrical] centre, i.e. the point of intersection of its diameters.

2 For any two bodies of equal force with parallel faces and equal heights, the ratio of their gravities is equal to the ratio of their volumes.
3 If a body with parallel faces is dissected by a plane parallel to its faces, it will be divided into two bodies with parallel faces as well. Each of them will have its own centre of gravity. Their centres of gravity will lie in the same straight line with the centre of gravity of the whole body, which lies in the dissecting plane. The ratio of heavinesses of the two bodies is an inverse of the ratio of the sections of that line.

(*Ibid.:* 20)

Al-Qūhī and Ibn al-Haytham restricted themselves to modifying and supplementing the Archimedean system of axioms to apply it to three-dimensional cases. As to al-Isfizārī, he goes further and builds up a theory of the centre of gravity of a system of three-dimensional bodies not connected rigidly with one another. He proceeds from the results of an experiment which consists of the following. Balls are let to roll into a hemispherical bowl: first one ball, then two balls of the same diameter and weight, and, finally, two balls of different diameters and weights (Figure 18.1). Thus, there is the centre of gravity of a single heavy body in the first case and that of a system of two bodies not connected rigidly with each other in the second and third cases. In the first case, the centre of gravity of the ball lies 'on the arrow' *(sahm)* which connects the centre of gravity of the bowl with the centre of the universe; in the second case, it is at the point of intersection of that 'arrow' with the straight line connecting the centres of gravity of the balls; in the third case, it is at a point of the 'arrow' which is spaced from the centres of gravity of the balls at distances inversely proportional to their weights (*ibid.:* 40).

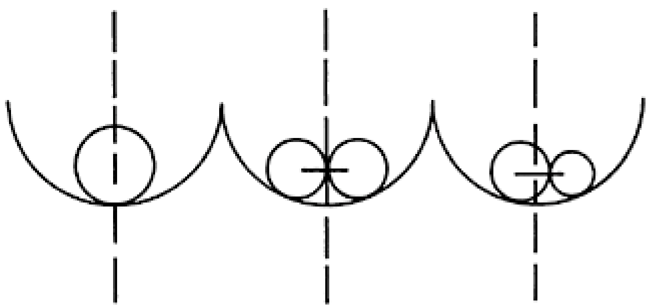

Figure 18.1

Al-Khāzinī first discloses the results of his predecessors and then determines the centre of gravity of a system of rigidly connected heavy bodies, considering lever scales (a system consisting of a balance beam, scales and loads) as an example of such a system. He first determines the centre of gravity of the free scales and then that of the loaded scales. It is characteristic that al-Khāzinī reduces the three-dimensional problem to a plane one (he passes on from bodies to plane figures) and finally to a problem on comparison of the areas of planes.

The development of the Archimedean tradition, however, was only one of the aspects of the teaching on centre of gravity in Arabic science. Alī the authors mentioned refer to a system of geometrical axioms, but at the same time postulate certain statements which combine Archimedean axiomatics with dynamic considerations. In their reasonings, the concept of the centre of gravity is associated with the concept of 'gravity' as 'force' and with the idea of the 'centre of the universe'.

Al-Khāzinī follows al-Qūhī and Ibn al-Haytham and formulates a number of statements, with two of them being of particular interest:

1 The point of a heavy body that coincides with the centre of the Universe when the body is at rest is called the centre of gravity of that body.
2 If the motion of a body comes to its end, the tendencies of all portions of the body to the centre of the Universe are the same.

(*Ibid.:* 17, 18)

The first definition is a classical example of amalgamation of the geometrical and dynamic traditions. The second statement is formulated in the dynamic spirit. But this, at the first glance purely dynamic 'spirit', has the Archimedean basis under it. Indeed, when speaking about the same 'tendency' of all portions of a body to the 'centre of the universe', al-Qūhī and Ibn al-Haytham actually have in view the Archimedean concepts of ῥοπή ('tendency') and of equal moments of force. As a matter of fact, the centre of gravity of a body is defined as a point in which the sum of the moments of the gravity forces acting on the body is equal to zero.

Al-Qūhī and Ibn al-Haytham formulated this system of axioms for a single heavy body. Al-Isfizārī spreads its application to systems of heavy bodies. He states that each heavy body tends to the centre of the universe. On its way to this centre, a body could meet an obstacle, say, another heavy body. Each of them will move to the centre of the universe and will touch each other in their motion so that they 'become, as it were, a single heavy body having a common centre of gravity' which approaches the centre of the universe (*ibid.:* 39). The centres of gravity of the two initial bodies turn out to be from the common centre of gravity at distances inversely proportional to their gravities.

'The existence of such a relationship [says al-Isfizārī] is the cause of rest of these two bodies, since the centre of gravity of each of them tends to the centre of the Universe in accordance with its force' (*ibid.:* 39).

The law of lever, the equilibrium of a system of bodies, stability of equilibrium

Statics, as the science of weighing, was based both in Antiquity and in the medieval East on the theory of the lever. The essence of the theory of the lever reduces to the problem of the equilibrium of a system of two bodies. Archimedes considered only a balanced weightless lever which he imagined in the form of a straight line section fastened at a certain point, with loads suspended from the ends of the section on weightless filaments. The Archimedean law of the lever is a direct consequence of the theory of the centre of gravity.

Another approach to the theory of the lever ascends to the kinematic tradition of *Mechanical Problems* which proceeds from an unbalanced lever. In that case, the proof of the law of the lever is based on the notion that, if the equilibrium of a lever is disturbed, its arm describes an arc whose length is inversely proportional to the magnitude of the suspended load.

The medieval authors have used both traditions. Both versions of the law of the lever can be found in one and the same treatise, for instance, in *Kitāb al-qarasṭūn* or *Kitāb mīzān al-ḥikma*.

In *Kitāb al-qarasṭūn*, the law of the lever is proved twice. In the first proof, Ibn Qurra proceeds from *Mechanical Problems*. His proof is essentially reduced to comparing the areas of two sectors described by the arms of an unbalanced ponderable lever and is not very strict. Actually, he considers a mechanical model of the phenomenon and gives it a geometrical interpretation. The second, more rigorous, proof ascends to the Archimedean tradition. It is the result of the application of the mathematical body of antique mathematics—the Eudoxus-Euclid theory of ratios and the Archimedean method of upper and lower integral sums—to problems of statics. Here, Ibn Qurra proceeds from the principal concepts of *The Book of Euclid on the Balance* and *Liber de canonio*.

In *The Book of Euclid on the Balance,* the principal law of the lever is only proved for commensurable loads and, on the face of it, only for a weightless lever. In the course of his proof, however, the anonymous author divides the lever into an arbitrary number of equal portions by suspending equal loads at division points and demonstrates that all these loads can be replaced by a single load equal to their sum and suspended in the middle of the lever arm, i.e. by their resultant. Thus, he goes on from a geometrical line to a ponderable lever.

The author of *Liber de canonio,* proceeding from what was proved by pseudo-Euclid, operates with the concept of a ponderable lever from the very beginning. He regards the lever as a ponderable beam of homogeneous material and constant thickness. In the course of the proof he regards the weight of a beam section as a load uniformly distributed along its length, and so it can be replaced by an equal load suspended from that section, assuming the section proper to the weightless.

Both these notions were used and developed by Thābit ibn Qurra.

Ibn Qurra analyses successively levers with commensurable and incommensurable loads, first a weightless lever and then a ponderable one. The problem of the equilibrium of a ponderable lever is reduced to determining the resultant of a continuous load uniformly distributed over a beam section, in other words, to determining the centre of gravity of a ponderable beam section.

In terms of mathematics, the problem is equivalent to calculating the integral $\int_a^b x\,dx$, i.e. it is equivalent to the problem of determining the volume of a paraboloidal segment, which was solved by Ibn Qurra in *Maqāla fī misāḥat al-mujassamāt al-mukāfiya (The Book on Measuring Paraboloidal Bodies)*. Ibn Qurra first determines the resultant of two equal forces, then generalizes the result obtained for any arbitrary number of equal forces and for an infinite number (*lā nihāya*—literally without end) of such forces, and finally goes on to a continuous load distributed uniformly on a beam. He gives a strict proof of the result obtained by using the Archimedean method of upper and lower integral sums (Rozhanskaya 1976:91–3).

Al-Khāzinī initially gives the classical Archimedean formulation and then abstracts from *Kitāb al-qarasṭūn* and from another treatise of Thābit ibn Qurra, 'Bāb mufrad fī ṣifāt al-wazn wa ikhtilāfihi' ('Separate chapter on the weight's property and its difference'), which has come to us only in al-Khāzinī's exposition *Kitāb mīzān al-ḥikma*.

Al-Khāzinī further discloses the theory of the lever according to al-Isfizārī. Al-Isfizārī first appeared in the history of statics when he proposed an explicit definition of a ponderable lever, which is worth citing in full.

> The sequence of logical conclusions which we have presented geometrically is based on the assumption that the balance beam is a certain imaginary line. It is known that an imaginary line has no weight. Balancing of loads is impossible on it. We cannot suspend from it something that we want to weigh since it is not a real line. But the balance beam…represents a heavy body and its proper weight may be the reason of disturbance of equilibrium if the point of suspension is not the mid-point of the beam.
>
> (al-Khāzinī, *Kitāb mīzān* ***al-ḥikma***, English translation, pp. 44–5)

As Ibn Qurra, al-Isfizārī combines two versions of the law of the lever: the Archimedean version in which his reasoning is close to the method of *The Book of Euclid on the Balance* and actually coincides with Ibn Qurra's proof, and the version coinciding with that in *Mechanical Problems*. Following the latter source, he states that 'lever scales can be reduced to a circle, since the portions of the balance beam at both sides from the suspension point are analogous to lines passing from the centre of a circle' (*ibid.:* 100) and the suspension point proper is the centre of the circle.

The motion of the ends of an unbalanced lever is associated with the Aristotelian concepts of 'natural' and 'forced' motion. The descending load on a balance performs a 'natural' motion, whereas the ascending load is in a 'forced' motion. According to al-Isfizārī, the cause of the 'forced' motion of one of the ends of a balance is not a 'force' or any other external action, but the 'natural' motion of the other end. In turn, the cause of this 'natural' motion is the 'natural inclination' of a heavy beam to the 'centre of the Universe'.

Thus, the condition of equilibrium of a lever is reduced by al-Isfizārī to the condition of equality of inclinations. 'A balance beam, [he says] will retain its equilibrium,…if the inclinations of the loads at its both ends are neither increased not decreased' (*ibid.:* 42).

The second portion of al-Isfizārī's proof matches pseudo-Euclid (up to introducing the concept of the 'force of weight') and to *Kitāb* ***al-qarasṭūn*** (up to replacing a load by a large number of smaller loads transferred into a single point and using the proof by contradiction).

Al-Khāzinī presented Ibn Qurra's and al-Isfizārī's proofs so exhaustively that it enabled him to leave the law of the lever proper, and pass on immediately to its practical applications. He considers the balance as a system of heavy bodies (the balance beam, pointer and scales with loads whose number may be up to five[2]). Then he studies the conditions of their equilibrium and stability on the basis of the theory of the centre of gravity which he had disclosed earlier.

The study is performed in a number of stages. Initially, he considers a heavy cylindrical beam (balance beam) suspended freely on an axis and in equilibrium parallel to the horizontal axis. Al-Khāzinī considers three probable positions of the beam on disturbance of its equilibrium, depending on whether the axis of rotation passes through, above or below the centre of gravity of the beam. He calls these positions respectively 'the axis of equilibrium' (*miḥwar al-iʿtidāl*), 'the axis of turning' (*miḥwar al-inqilāb*) and 'the axis of constraint' (*miḥwar al-iltizām*). By modern terminology, these are respectively cases of indifferent, unstable and stable equilibrium.

Al-Khāzinī's characteristics of these cases are as follows:

> First case: the axis of equilibrium.
>
> If the axis passes through the centre of gravity of the balance beam (which is located in the mid of the beam) and is perpendicular to the beam, the latter will rotate freely obeying its own gravity and will remain at rest in the position in which it has been stopped during rotation by the gravity. The beam acquires the horizontal position under the action of gravity, since, when it is stopped, the arrow passing from the centre of the Universe to the centre of gravity (of the beam) divides the beam into two equal portions.
>
> Second case: the axis of turning.
>
> Let the axis be now located between the centre of the Universe and the centre of gravity of the balance beam. If the balance beam is now put into motion, it will turn over, since the arrow passing from the centre of the Universe divides it into two unequal portions, the larger portion outweighs the smaller one, and the beam turns over.
>
> Third case: the axis of constraint.
>
> This is when the axis of rotation of the balance beam is above its centre of gravity. If the beam is caused to move in this case, the arrow passing from the centre of the Universe to the centre of gravity will divide the beam into two unequal portions. The larger portion will turn upwards and outweigh the smaller one and then will move downward and stay parallel to the horizontal, since now the arrow divides the balance beam into two equal portions. This constrains the beam to stay parallel to the horizontal.
>
> (*Ibid.:* 97–8)

At the second stage of his analysis, Al-Khāzinī considers the system consisting of the balance beam and pointer and disregards, for the time being, the effects of scales and weights. The equilibrium conditions for such a system can be reduced to the equilibrium of a free balance beam, but having a different centre of gravity. Besides, these considerations are true provided that the system is symmetrical relative to the axis of suspension, i.e. the balance pointer has a rhombic shape and is fastened in the centre of symmetry of the balance beam. Al-Khāzinī gives geometrical illustrations for these two stages of his analysis (see Figures 18.2 and 18.3). If these

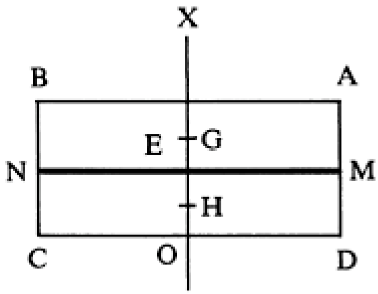

Figure 18.2

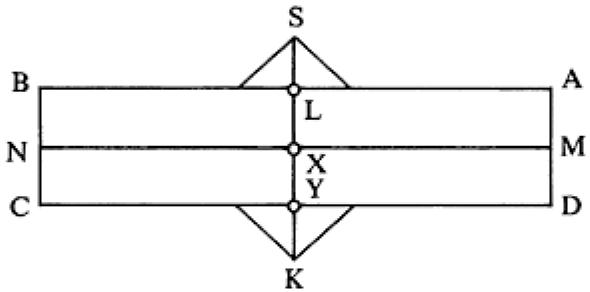

Figure 18.3

conditions are not observed, i.e. if the pointer has a different shape and is fastened not in the centre of symmetry and not on the axis of symmetry, the centres of gravity of the balance beam and pointer will coincide neither with each other nor with the point through which the axis of rotation of the beam passes. The conditions of equilibrium for this case become more complicated, even more so when scales are suspended from the balance beam. Al-Khāzinī gives no proof of this statement, referring only to this being 'too extensive'. His method, however, enables us to suppose that his 'extensive proof' was based on certain statements in Archimedes' treatise *On the Floating Bodies,* in particular, on the stability of equilibrium of bodies of various shapes immersed into liquid. If this is the case, al-Khāzinī was evidently familiar not only with the Arabic treatment of that work, which was given completely in *Kitāb mīzān al-ḥikma* (but which contained no statements on the stability and instability of bodies immersed into liquid), but also with the original Greek text which became known to European science only at the beginning of our century.

Hydrostatics

Hydrostatics in the medieval East was also founded on the Archimedean tradition. Scientists of that time were familiar both with Archimedes' treatise proper *(On the Floating Bodies)* and with its treatments, such as *The Book of Archimedes on the Heaviness and Lightness* which has been mentioned earlier, Menelaus' treatise and al-Kindī's *Great Treatise on Bodies Which are Immersed in Water,* the most detailed treatment of Archimedes' work (see Wiedemman 1970b:160).

These data are presented in the most concise form by al-Khāzinī, who combines the Archimedean hydrostatics with the Aristotelian teaching on motion of bodies in a material medium. The principle by which al-Khāzinī selected sources for the respective chapter of *Kitāb mīzān* ***al-ḥikma*** is clear. He includes his treatments of Archimedes' and Menelaus' works to disclose the fundamentals of hydrostatics. By including *The Book of Euclid on the Heaviness and Lightness* into his treatise, he acquaints the reader with the teaching on motion of bodies in material media.

'If a heavy body moves in a liquid, the heaviness of the body will decrease by a magnitude depending on its volume until it becomes lighter in the liquid by the magnitude of weight in a unit of volume…' (al-Khāzinī, *Kitāb mīzān* ***al-ḥikma***, English translation, p. 24). The larger is the volume of a moving body, the greater is the hindrance (i.e. the buoyancy force).

On the other hand, the different velocity of motion of two bodies in a liquid, which are equivalent in their volume and density, is determined by their different shapes. 'The force of motion of various bodies in air or water may be different. The cause of this difference is in their different shapes' (*ibid.:* 24).

Thus, al-Khāzinī distinguishes two kinds of forces acting on bodies moving in a material medium. One of them, which offers resistance to the motion in accordance with the Aristotelian teaching, is determined by the weight and shape of a body.

Another, Archimedean, force is associated with the volume of the body proper and of the liquid displaced by it, and depends additionally on the density of the medium.

If two bodies have the same volume but different density, the body having a higher density has a greater heaviness in a given medium. Bodies made of the same substance and having the same heaviness in a particular medium, may have different weights in another medium.

These statements undoubtedly correspond to the Archimedean teaching. Actually, al-Khāzinī applies Statement VII of the first book of the treatise *On the Floating Bodies* to bodies placed into media of different density, and not only into water.

Thus, having combined the Archimedean hydrostatics and the Aristotelian theory of motion of bodies, al-Khāzinī develops a unified theory for a general case of motion of a body in a liquid which considers both the resistance of the body and medium and the buoyancy force.

Of special interest are his discourses on variations of the weight of bodies transferred from one medium into another (for instance, from water into air or vice versa). They served as a theoretical substantiation of his method for determining the specific weight by weighing a specimen successively in air and in water.

Al-Khāzinī spreads the Archimedean hydrostatics, i.e. the teaching on continuous bodies floating in a liquid, to floating 'bodies with a cavity', i.e. he develops the theory of a ship. He simulates a ship as a body with an open cavity which is immersed into liquid, and a loaded ship as a similar body with a load placed in the cavity.

Al-Khāzinī divides his discourse into three stages. He considers first a continuous body immersed into liquid, then a free 'body with a cavity' and finally a loaded 'body with a cavity'.

Using a number of definitions, he reduces the model of a loaded 'body with a cavity' to a free 'body with a cavity' and this, in turn, to a free continuous body, i.e. the theory of floating of a loaded ship is reduced by him to the Archimedean theory of bodies floating in a liquid (*ibid.:* 27–8).

APPLIED STATICS

Practical, or applied, statics, as it is understood now, was in the medieval East the subject of a number of scientific disciplines which, according to the classifications of sciences existing at that time, were related to different 'sciences' and their 'branches', so that the interrelations between them could not always be established. Geometrical statics was considered a division of geometry; standing apart from 'the science of weights' which is now referred to as theoretical statics. Applied statics proper included what was called *'ilm al-ḥiyal,* the teaching on simple machines and their combinations. It is sometimes seen in the works of that period, as with antique authors, that mechanics was subdivided into the science of military machines and the science of 'ingenious devices' (*al-ḥiyal*), the most important of which were mechanisms for lifting loads and for water irrigation.

Now applied statics is initially regarded as a complex of problems related to *'ilm al-ḥiyal,* i.e. to mechanics in its original, narrower sense. The theory of the balance (as a variation on the theory of the lever) and the theory of weighing are subdivided into the theory of simple machines and their combinations. The theory of weighing is closely associated with the problem of determining the specific weight. This problem was separated quickly into a special and very important branch of practical statics, which attracted the interest of many outstanding Arabic scientists.

The theory of simple machines and mechanisms (*'ilm al-ḥiyal*)

Among numerous treatises on *'ilm al-ḥiyal,* we shall choose those whose authors consider devices based on the application of the 'golden rule of mechanics'. Among such mechanisms, special interest was paid to those intended for load lifting. Descriptions of various modifications of simple machines are as a rule found in any encyclopaedia of that time.

One of the earliest Arabic sources dealing with 'simple machines' is Abū 'Abd Allāh al-Khwārizmī's *encyclopaedia Liber* *mafātīḥ al-'ulūm* *(The Keys of Sciences),* which was found in medieval Europe as a Latin translation. Al-Khwārizmī's encyclopaedia contains descriptions of devices by means of which 'large loads could be moved by a small force'. Most of these mechanisms were mentioned in Hero's *Mechanics*.

Of greatest interest in that sense, however, are the works of Ibn Sīnā, in particular the chapters on mechanics in his encyclopaedic books *Mi'yār al- 'aql* *(The Measure of Mind),* which are founded on *Mechanical Problems* and Hero's *Mechanics*.

This treatise, which consists of two sections, is specially devoted to the description of five simple machines. In the first section, Ibn Sīnā follows Hero very closely and even borrows the descriptions and drawings of some 'simple machines' from Hero's

Mechanics. The structure of the first section is also largely due to Hero's treatise: the names and definitions of 'simple machines', their material requirements and the conditions to ensure their stability and reliability.

The second section of the treatise contains descriptions of combinations of 'simple machines'. Like Hero, Ibn Sīnā classifies these combinations and groups by the principle of likeness or unlikeness of the constituent 'simple machines'. However, in contrast to Hero who considered only some of these combinations, Ibn Sīnā successively analyses all probable combinations. Initially he describes all combinations of like 'simple machines', levers, pulleys, windlasses, screws, similar to Hero. After that, Ibn Sīnā considers all practically probable pairwise combinations of unlike 'simple machines': windlass-screw, windlass-pulley, windlass-lever. Finally, he describes a mechanism which is essentially a combination of all 'simple machines' (except for the wedge).

Though Ibn Sīnā's treatise is a purely practical manual, it has great significance in the history of mechanics. In fact this was the first successful attempt to classify simple machines and their combinations. It should be noted that the interest in such a classification was by no means occasional both for Ibn Sīnā himself and for his epoch. Arabic science of that period can generally be characterized by a tendency to classify the available scientific evidence. It is well known that after al-Fārābī, Ibn Sīnā himself wrote a treatise on the classification of sciences. It may be stated that Ibn Sīnā's treatise on mechanics crowned the period of assimilation of the antique scientific heritage in the field of applied mechanics and marked the beginning of a new stage of its development.

This stage, which belongs chronologically to the eleventh to twelfth centuries, can be characterized by an utterly different tendency. Authors of that time usually consider a single particular kind of 'simple machine', disclose as strictly as possible its theory, and then give descriptions and classifications of various devices which are modifications of the kind considered. Or else they consider a particular subdivision of a 'branch' of science and again describe diverse 'machines', mechanisms and instruments which are based on that branch or associated with it. A typical example of this type of mechanical treatise is al-Khāzinī's *Kitāb mīzān al-ḥikma*, which discloses exhaustively the principal problems of the theory and practical applications of the commonest kind of 'simple machine', the lever, and of its most popular modification, the balance.

Thus, the science of 'simple machines' in the epoch of Antiquity and in the medieval East passed through a number of characteristic stages of evolution: from the initial descriptions of the operating principle of 'simple machines' and their combinations to attempts to classify them and then to monographic descriptions of individual kinds of machine, which contained the theory, design and all modifications of a particular kind of machine. Such were the characteristic features of that stage in the development of statics, from which engineering mechanics has originated (Rozhanskaya and Levinova 1983:101–14).

The balance and weighing

As was indicated earlier, the most exhaustive information on the theory of balance and weighing is contained in al-Khāzinī's *Kitāb mīzān al-ḥikma*. The author himself characterizes his book in the following words: 'This is all what has been accumulated on balances and methods of weighing on them' (al-Khāzinī, *Kitāb mīzān al-ḥikma*, English-translation, p. 7).

Al-Khāzinī divides all varieties of balance into two groups: equal-arm and unequal-arm balances. The simplest design of an equal-arm balance has a balance beam with scale pans; a load is placed onto a scale pan and weighed by means of balance weights placed on one or both pans. For this type of balance Al-Khāzinī suggests a set of balance weights which make it possible to weigh the maximum load with the least number of weights. The important point is that the masses of the weights are chosen as powers of two and three, i.e. they are equal to 1, 2, 2^2, 2^3,…3, 3^2, 3^3…etc. weight units. In that association, al-Khāzinī gives the solution of the 'problem on weighing' which was later well known in medieval Europe and whose roots are found in oriental mathematics (Rozhanskaya 1976:124–8).

Unequal-arm balances are subdivided by al-Khāzinī into two types: the *qarasṭūn*, i.e. a balance with two scale pans or hooks for suspending the loads, and the *qabbān,* i.e. a balance with one scale pan and a balance weight which can be moved along the other arm. The theory of these types of balance was disclosed in the treatments of Ibn Qurra's and al-Isfizārī's works which were incorporated by al-Khāzinī into *Kitāb mīzān al-ḥikma* (see English translation, pp. 33–51).

From the application standpoint, al-Khāzinī divides both types of balance into a number of varieties. He names the varieties of *qabbān* as *qusṭās mustaqīm,* the 'right balance' for especially precise weighing, and the astronomical timepiece balance. He then describes the different versions of the *qarasṭūn*: the change balance whose beam is divided in a ratio 10:7 (the dinar-to-dirham ratio), the equal-arm 'soil balance' for geodetic work and, finally, a large group of 'water balances' (hydrostatic balances) intended for weighing specimens of metal and minerals in air and water to determine the specific weight and composition of alloys. Al-Khāzinī gives special attention to this group of balances. An essential portion of his treatise is devoted to methods of weighing metals and minerals in water in order to determine their specific weights.

Al-Khāzinī divides all water balances into three types. The first type is a simple, usually equal-arm, balance with two scale pans. The second type has three pans, two of which are suspended one beneath the other, and is used for weighing in water. The third type has five pans, with three of them fastened at the ends of the balance beam, as in the previous type, and the two other pans being movable along the beam for balancing.

Al-Khāzinī gives a detailed account of the history of evolution of the water balance and the methods of weighing over the previous fifteen centuries, beginning from the water balance of Antiquity up to a balance of his own design, and estimates the contributions of all the scientists mentioned by him to the theory of balances and the practice of weighing.

The improvement of the water balance resulted in the appearance of a third pan specially for weighing specimens in water. According to al-Khāzinī, a water balance with three pans was employed already by some of his predecessors in Islamic countries.

Al-Isfizārī increased the number of pans to five and constructed a universal balance which he called 'the balance of wisdom'. The balance of wisdom is essentially an equal-arm balance with two dials, five hemispherical pans, a movable weight and a pointer fastened at the mid-point of the balance beam. The balance was connected to a stand not by an axle but by a clever free suspension, a combination of a cross-bar and a ⊓-shaped piece called a 'shear' which evidently was designed by al-Isfizārī himself. This suspension minimized the effect of friction on the sensitivity of the balance of wisdom. The high sensitivity was also ensured by proper selection of the dimensions of the balance beam and pointer, the angle of bending of the balance beam, the sharpness of the pointer etc. The description of the balance and its parts, the method of assembly and the problem of its equilibrium and sensitivity occupies a whole chapter in *Kitāb mīzān al-ḥikma*. Two fixed pans of the balance were intended for weighing in air, and the third fixed pan for weighing in water. The two movable pans and the movable weight served as riders to bring the balance to equilibrium before calibration and weighing (Figures 18.4 and 18.5).

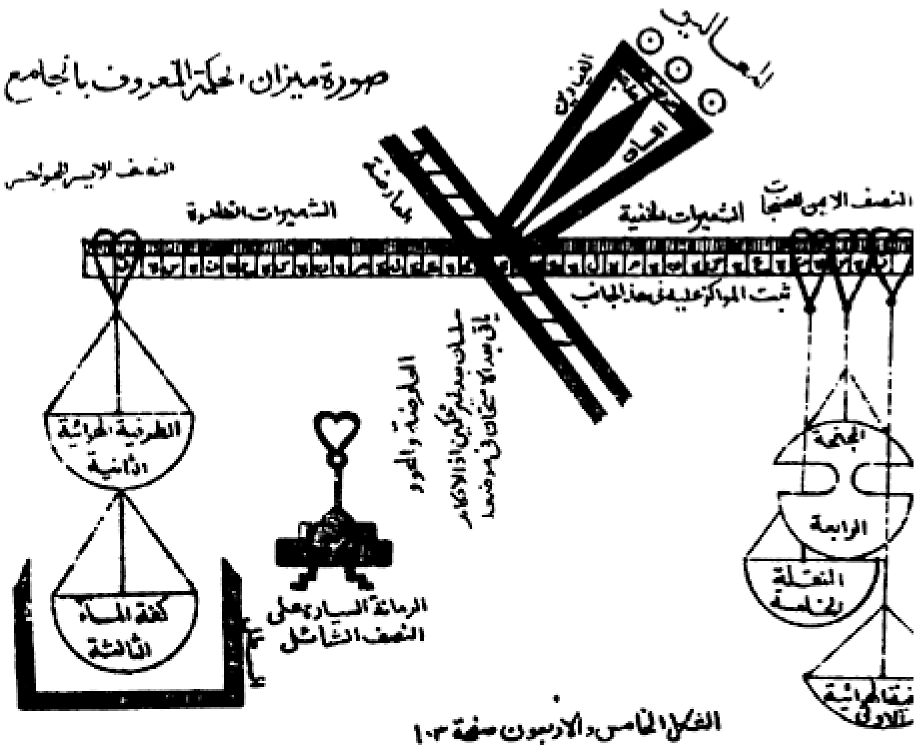

Figure 18.4

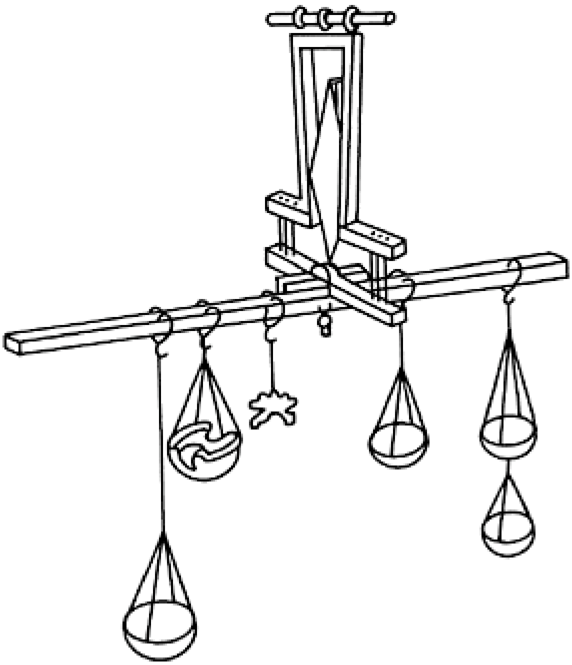

Figure 18.5

The ‘balance of wisdom’ was later improved by al-Khāzinī who also developed its theory and the methods of its calibration and experimental weighing.

Al-Khāzinī describes in detail the method for determining the ‘water weight’ of a specimen, in which an essential point was the calculation of the buoyancy force.

The ‘balance of wisdom’ was calibrated in the following manner. Before operation, the balance was set to equilibrium with the water pan immersed in water. A specimen of a known weight was then placed onto the left-hand fixed pan and counter-balanced by placing weights onto the right-hand fixed pan. The specimen was then transferred into the ‘water pan’ and the balancing weights into the right-hand movable pan, then the balance was brought to equilibrium by moving the movable pans along the balance beam on both sides of the axis so that the pans always remained at equal distances from the axis. The point at which the movable pan with weights balanced was called the ‘centre’ of a suspension (metal or mineral), i.e. the point corresponding to the specific weight of that substance.

Special requirements were specified concerning the quality of the specimens and the physico-chemical properties of the water. As indicated by al-Khāzinī, experiments should be carried out only with water from a particular source and at a particular constant temperature of air.

The ‘centres’ of metals and minerals on the scale of al-Khāzinī’s balance can be arranged in the following order of decreasing specific weight. Metals: gold, mercury, lead, silver, bronze, iron and tin; minerals: sapphire, ruby, spinel, emerald, lapis lazuli, rock crystal and glass.

Al-Khāzinī points out that the balance can be set to equilibrium only at one point. Therefore, the specific weight of a given substance and the composition of a given alloy are determined uniquely. If the equilibrium of the balance can be attained at a few points, this means that the specimen is an alloy of three or more components. In that case, the problem cannot be solved unambiguously.

In addition to calculating the specific weight and composition of alloys, the 'balance of wisdom' could be used for proving the authenticity and purity of metals and minerals and for some other purposes. It was considered the most perfect of the water balances that were known in the twelfth to thirteenth centuries.

The importance of the 'balance of wisdom' in the history of balances and weighing is also due to its all-purpose application. With only two pans and the movable weight left on the balance beam, it could be used as a common *qarasṭūn* or *qabbān,* as a 'dirham-to-dinar' changing balance, as an especially precise 'right balance' *(qusṭās mustaqīm),* etc. Thus, it was indeed a highly sensitive instrument with an unusually wide range of applications.

Specific weight

The available data on the first attempts to determine the specific weight are very scarce. The earliest among them ascends to the wide-spread legend about Archimedes who determined the composition of the alloy from which the crown of Hiero, the tyrant of Syracuse, was cast. It is also known that Menelaus of Alexandria worked on the problem.

The studies of the problem of specific weight in Arabic science are summarized in two principal sources: al-Bīrūnī's treatise on specific weights *(Maqāla fī al-nisab)* and *Kitāb mīzān al-ḥikma* mentioned earlier. (It should be noted that al-Bīrūnī's treatise was included by al-Khāzinī almost completely as one of its part (English translation pp. 55–78).) From al-Bīrūnī and al-Khāzinī we can gain some knowledge of the studies undertaken by the scientists in Islamic countries: e.g. Sind ibn ʿAlī (ninth century) and Yuḥannā ibn Yūsuf (tenth century), who belonged to the Baghdad school, Abū al-Faḍl al-Bukhārī (tenth century), whom al-Bīrūnī considered his direct predecessor, al-Nayrīzī (tenth century), al-Rāzī (tenth to eleventh century), ʿUmar al-Khayyām (eleventh to twelfth century) and others.

It should be emphasized, however, that the specific weight as the ratio of the weight of a body to its volume was not strictly defined in Antiquity nor by al-Khāzinī's predecessors in the Islamic countries. Alī predecessors of al-Khāzinī whom he mentioned and who were named by al-Bīrūnī in the introduction to his treatise, actually made use of the concept of specific weight but had not defined it explicitly. It was al-Khāzinī who produced the first strict definition of specific weight:

> The magnitude of weight of a small body of any substance is in the same ratio to its volume as the magnitude of weight of a larger body (of the same substance) to its volume.
>
> (*Kitāb mīzān al-ḥikma,* English translation, p. 86)

In order to determine the specific weight of a specimen, one had to know the weight of that specimen in air and water and the volume and weight of the water displaced by the specimen. For that reason, water balances played an important part in such experiments and were employed by most researchers. In particular, al-Bīrūnī himself designed an ingenious instrument to determine the volume of displaced liquid. To determine the specific gravity he used a 'conical vessel' to find the ratio of the weight of water displaced to the weight of a substance in air.

Once these data were obtained, one could easily calculate the specific weight by simple mathematical procedures. Al-Bīrūnī carried out a series of measurements of specific weights. He took specimens of metals and minerals of equal weight (100 mithqāls; 1 mithqāl=4.424 g) or equal volume (the volume occupied by 100 mithqāls of gold). He summarized the results obtained in a number of tables: a table of the weight of water displaced by specimens of metals and minerals which had the same weight in air; a table of the volume of specimens having the same weight in water etc. The specific weight could be found from these tables arithmetically. As a reference substance, al-Bīrūnī did not take water, as is adopted at present, but the heaviest metal gold for metals and the heaviest mineral sapphire for minerals.

Al-Bīrūnī's results are quite close to the modern data. Some deviations can be explained by the impurity of specimens and by temperature differences in his experiments (al-Bīrūnī neglected the temperature of water).

The data presented by al-Bīrūnī can be easily recalculated from the gold or sapphire reference base to water. For re-calculation, it is sufficient to multiply the figure obtained by al-Bīrūnī by the ratio of the specific weight of the reference substance to that of water (3.96 for sapphire and 19.05 for gold) and divide by 100 (100 mithqāls is the weight of a specimen).

Al-Bīrūnī also determined the specific weight of some liquids, established the differences in the specific weights of hot and cold and fresh and salt water and pointed out a certain correlation between the density and specific weight of water. He evidently used an instrument with a special scale for liquids for these experiments, of the type of aerometer which was described by al-Khāzinī. Al-Bīrūnī was the first in the history of science to introduce checking tests into the practice of experiments.

'Umar al-Khayyām devoted a special treatise to the problem of determining specific weight. His work *Mīzān al-ḥikam (The Balance of Wisdoms)* was completely included in al-Khāzinī's treatise (pp. 87–92). He had used as a point of departure the relations of the weights in air and water. Khayyām proposed two calculating methods to determine the specific weights: by means of the theory of ratios and by an algebraic method called 'al-jabr wa-l-muqābala' (i.e. by means of the restoration and opposition), which reduce to modern canonical methods of solving linear equations. Khayyām determines the specific weight proceeding from the ratio of the weight of a substance in air to its weight in water. If P, P_1, P_2 and Q, Q_1, Q_2 are the weights of an alloy and its constituents respectively in air and water and d, d_1, d_2 are the corresponding specific weights, one can compare pairwise the ratios P/Q, P_1/Q_1 and P_2/Q_2 which are equivalent to the ratios of specific weights $d/(d-d_{water})$, $d_1/(d_1-d_{water})$ and $d_2/(d_2-d_{water})$.

Khayyām illustrates obtained proportions by a geometrical diagram in which numerical values are represented by sections of different length.

An essential contribution to the theory and practice of determining the specific weight was made by al-Khāzinī who devoted to the problem a substantial portion of *Kitāb mīzān* *al-ḥikma*. Having described in detail the methods used by the ancients (al-Bīrūnī and Khayyām), al-Khāzinī discloses his own method which was based on weighing on the balance of wisdom and using al-Bīrūnī's tables. Employing the 'balance of wisdom', al-Khāzinī obtained the weights of tested specimens (e.g. gold, silver and their alloys) in water and in air and used them to determine the specific weights of substances by the following three methods:

1 arithmetically, by means of the Euclidean theory of ratios and compiling the corresponding proportions;
2 geometrically; and
3 by means of 'algebra and *al-muqābala*', i.e. by solving first-order algebraic equations.

If, as given earlier, P, P_1, P_2 and Q, Q_1, Q_2 denote the weights of an alloy and its constituents in air and water, F, F_1, F_2 are their Archimedean forces, c is the value of division of the scale and m, m_1, m_2 are the numbers of scale divisions for the alloy and its components, al-Khāzinī's arithmetic method can be reduced to the equation

$$x = \frac{P(Q_2 - Q)}{Q_2 - Q_1} = \frac{P(m_2 - m)}{m_2 - m_1}$$

where $F=P-Q=cm$, $F_1=P_1-Q_1=cm_1$, $F_2=P_2-Q_2=cm_2$ and x is the weight of a component of the alloy.

Another, geometrical, method is as follows. Al-Khāzinī draws two parallel lines EG and HF and lays on them, on a definite scale, the following sections: EG=P, the weight of the alloy in air; $LF=Q$, its weight in water; HF=$\xi_1=PQ_1/P_1$, the weight of the gold fraction of alloy in water; and KF=$\xi_2=PQ_2/P_2$, the weight of the silver fraction of alloy in water (see Figure 18.6). He then draws straight lines EH and GK and extrapolates them to intersect at point X. They will inevitably intersect, which can be easily proved.

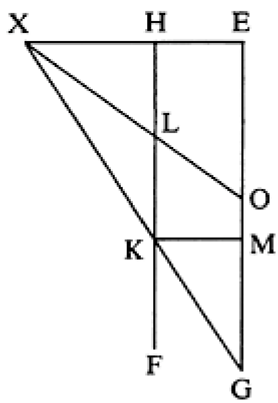

Figure 18.6

Let KM be drawn parallel to HE. Then the figure MEHK obtained is a parallelogram in which the sum of angles GEH and EMK is equal to two right-angles and the angle EXK is acute. Since, however, EMK is an exterior angle of the triangle MGK, the angle EGX is also acute.

Al-Khāzinī then draws a straight line XL at a certain angle to XHE, which intersects the section EG at a point O. In the general case, this point divides that section into two unequal portions. The point L is chosen so that LF=Q. If XO passes above EMG, the specimen being tested consists of pure gold; if it passes below XKG, the specimen is pure silver; and if it intersects the line XHE, the specimen is an alloy of these metals. The sections EO and OG are proportional to the percentage concentrations of these metals in an alloy.

Al-Khāzinī was the second among the authors known to us who employed the geometrical method. The first, as was mentioned earlier, was Khayyām. Khayyām's method, however, can be regarded merely as a geometric illustration of an arithmetic technique, whereas al-Khāzinī proposed a detailed and strictly proved geometrical method for solving the problem of mixtures and his diagram can be regarded a prototype of nomograms.

The third method proposed by al-Khāzinī is algebraic. Using the destinations given above, it can be represented as follows. The equation that was formulated by him in verbal form can be written as

$$Q = x\,\frac{Q_1}{P_1} + (P - x)\,\frac{Q_2}{P_2}$$

where Q_1/P_1 and Q_2/P_2 are the weight fractions of the constituents in the alloy and x is the sought-for weight of one of them.

Making the required procedures of 'restoration and opposition', this equation can be transformed to

$$x\left(\frac{Q_1}{P_1} - \frac{Q_2}{P_2}\right) = P\left(\frac{Q}{P} - \frac{Q_2}{P_2}\right)$$

whence we have

$$x = \frac{P(Q/P - Q_2/P_2)}{Q_1/P_1 - Q_2/P_2}$$

or

$$x = P\,\frac{Q - \xi_2}{\xi_1 - \xi_2},$$

i.e. the algebraic solution gives the same result as those obtained arithmetically and geometrically.

CONCLUSION

We have discussed above the process of the creation of the theoretical fundamentals and the development of the practical methods of Arabic statics.

This process was not reduced merely to the translation and compilation of antique works. First, the methods of Archimedes and of *Mechanical Problems* were improved and deepened in the ninth to fourteenth centuries; second, the dynamic aspect of Aristotle's teaching was developed further during that time.

Using a whole body of mathematical methods (not only those herited from the antique theory of ratios and infinitesimal techniques, but also the methods of the contemporary algebra and fine calculation techniques), Arabic scientists raised statics to a new, higher level. The classical results of Archimedes in the theory of the centre of gravity were generalized and applied to three-dimensional bodies, the theory of ponderable lever was founded and the 'science of gravity' was created and later further developed in medieval Europe. The phenomena of statics were studied by using the dynamic approach so that the two trends—statics and dynamics—turned out to be inter-related within a single science, mechanics.

The combination of the dynamic approach with Archimedean hydrostatics gave birth to a direction in science which may be called medieval hydrodynamics.

Archimedean statics formed the basis for creating the fundamentals of the science on specific weight. Numerous fine experimental methods were developed for determining the specific weight, which were based, in particular, on the theory of balances and weighing. The classical works of al-Bīrūnī and al-Khāzinī can by right be considered as the beginning of the application of experimental methods in medieval science.

Arabic statics was an essential link in the progress of world science. It played an important part in the prehistory of classical mechanics in medieval Europe. Without it classical mechanics proper could probably not have been created.

NOTES

1 In antique beliefs, the centre of the universe was coincident with the centre of Earth.
2 In 'the balance of wisdom', i.e. equal-arm lever balance with five scales and a balancing weight moved over the balance dial.

19
Geometrical optics

ROSHDI RASHED

INTRODUCTION

Arabic optics is derived from Hellenistic optics and, one could even say, only from it. Arabic optics has borrowed from Hellenistic optics its questions, its concepts, its results, and even the different traditions into which it divided in the Alexandrian era. One could say that the first Arab scholars who worked on optics joined the school of Hellenistic authors—Euclid, Hero, Ptolemy, Theon, and plenty of others—and them only. Optics distinguishes itself for this reason from other sectors of Arabic mathematical sciences, astronomy for example, in the sense that it received no other legacies, non-Hellenistic, so minute was it, that pushed with some weight on its development.

This very narrow dependence, however, did not hold up the relatively precocious emergence of innovative research. Very quickly after the massive transmission of Greek writings, the history of the discipline became that of the rectification of these writings, the accumulation of new results and the renewing of the principal chapters. Two centuries sufficed to prepare for what was finally a true revolution, which marked for ever the history of optics, indeed, more generally, that of physics. It is this dialectical movement between a solid continuity and a profound rupture that it is our duty to describe, in order to grasp the progress of Arabic optics between the ninth and the sixteenth centuries.

Let us place ourselves in the ninth century, in the mid-ninth century, more exactly. The Arabic translations of Greek texts on optics appear side by side with the first researches written directly in Arabic in this discipline. This simultaneity—which has not been sufficiently emphasized—of translation and research is not the privilege of optics; it can be seen through all the mathematical disciplines, if not for the totality of ancient heritage. The simultaneity is for us of major importance if we want to understand the nature of the movement of the translation and elaboration of optics. Never passive, the translation seemed on the contrary linked to the most advanced research of the era. Even if the names of the translators of the optical writings and the exact dates of the translations have not reached us, we know that these works of translation took place, in the majority, during the first half of the ninth century. The accounts of translators and scholars such as Qusṭā ibn Lūqā and Ḥunayn ibn Isḥāq, of scholar-philosophers such as al-Kindī—all in the ninth century—of ancient bibliographers such as al-Nadīm, do not allow us to go back surely and effectively beyond that century for the collection of writings on optics, with the exception of some vestiges relating exclusively to ophthalmology.[1] But the reading of the scholars of the ninth century—Ibn Lūqā or al-Kindī—reveals a knowledge of the Arabic version of *Optics* by Euclid, of that of Anthemius of Tralles, among others.[2] These translations cover the ensemble of areas of Hellenistic optics: optics in its proper sense, i.e. the geometrical study of perspective as well as the illusions which are linked to that; 'catoptrics', i.e. the geometrical study of the reflections of visual rays on mirrors; burning mirrors, i.e. the study of the convergent reflection of solar rays on mirrors; atmospheric phenomena such as the halo and the rainbow. These are precisely the areas of optics which are recorded slightly later by al-Fārābī in his *Enumeration of the Sciences*.[3] To these geometrical subjects it is necessary to add on the one hand the expositions on the theory of vision which feature in the ophthalmological works of physicians just as in the works of philosophers and, on the other hand, among the latter again, reflections on the theories of physical optics—colours for example.

A scholar from the middle of the tenth century thus had use of the translation of the *Optics* of Euclid and the major part of the *Optics* attributed to Ptolemy.[4] He had a more or less indirect access to the *Catoptrics* of Pseudo-Euclid and to certain writings in the tradition of Hero of Alexandria. He also knew almost all of the Greek writings on burning mirrors, of which some have not survived anywhere other than in their Arabic version. Besides a compilation of the book by Diocles, writings by Anthemius of Tralles, of a certain Didymus and of a Greek author who has still not been identified and is referred to as 'Dtrūms'[5] have been put into Arabic. He could also read in their Arabic translation the *Meteorologica* by Aristotle,[6] and certain of commentaries on it such as the one by Olympiodorus.[7] He was informed, at least of their contents, about the works of Galen on the anatomy and physiology of the eye.[8] He had the use finally of the compositions of philosophers on other aspects of physical optics, such as the one by Alexander of Aphrodisias on colours.[9]

This massive movement of translation of optical texts was not provoked, as one would be tempted to think, by only scientific and philosophical interests, but also by anticipated applications. Caliphs and princes encouraged research into what scientists had presented to them as a formidable weapon, and had allowed Archimedes to defeat the fleet of Marcellus: burning mirrors.[10] It is for these princes that research into catoptrics was taken up, to fill them with wonder and to entertain them.[11] These two types of application were not new: they were already known in the antiquity.[12]

Let us recall the first essays in Arabic, contemporary, we said, of these translations. There were at first the ophthalmological works, of which certain were composed before all the other contributions to the subject of optics—the first writings on the eye appeared in the eighth century. These works have spread with Ibn Māsawayh, and above all with Ḥunayn ibn Isḥāq, Qusṭā ibn Lūqā and Thābit ibn Qurra. We examine later the contribution of this medical school to physiological optics. We keep here to other aspects of optics.

According to the ancient bibliographers, two contemporaries led research in optics: Qusṭā ibn Lūqā and Abū Isḥāq al-Kindī. Only one treatise is attributed to the first, dedicated to burning mirrors; it is definitely a composition of the eminent translator and scholar, and not a translation of a Greek writing, as emphasized by the tenth-century bibliographer al-Nadīm. If it existed, this treatise has nevertheless not reached us, while a treatise attributed to the same author, which the bibliographers do not mention, has survived.[13]

Associated with the name al-Kindī are four memoirs on optics and catoptrics, three on burning mirrors and their construction, and three on physical optics.[14] Is that an exact inventory, or was there duplication of titles?[15] To this question, we cannot give a rigorous response. We only know that from the first group there remains the Latin translation of one of his books on optics, known under the title *Liber de causis diversitatum aspectus* (named by *De aspectibus*) and a critical commentary on Euclid's *Optics*[16]; of the second, an important treatise on burning mirrors has reached us,[17] and finally, from the third group, two writings. Be that as it may, with Qusṭā ibn Lūqā and above all al-Kindī, we witness the dawning of optical and catoptrical Arabic research.

THE BEGINNINGS OF ARABIC OPTICS: IBN LŪQĀ, AL-KINDĪ AND THEIR SUCCESSORS

The translation into Arabic of *Optics* by Euclid and the transmission of a part of the contents of the *Catoptrics* of Pseudo-Euclid are at the origin of numerous writings with different intentions and scope: new applications, new works where one amends, indeed rectifies, certain points of the *Optics* by Euclid. But, to this Euclidean tradition are added, in the ninth century already, that of Hero of Alexandria, who seems to have been known relatively early, that of catoptricians who were interested in burning mirrors, and that of philosophers and notably of Aristotle. This multiplicity of sources seems to be at the origin of the first project by scholars in the ninth century. For one of the main traits of this project is precisely the amendment of the *Optics* by Euclid.

One of the first books of Arabic optics is, we have said, that attributed to Qusṭā ibn Lūqā, recently discovered and never before analysed.[18] In this book, Qusṭā ibn Lūqā names the science, delimits its object and reveals to us his concept of its status.

Two terms interact in effect to define this discipline: 'the science of the diversity of perspectives' and 'the science of rays'. These are the two terms that al-Kindī chose also. From the first term, we have retained the term 'perspectives'—*manāẓir*—which is ἡ ὀπτική Such was already the situation in the ninth century, as one can read from the pen of Thābit ibn Qurra.[19] As regards the object of the science, it is no other than the study of this diversity of perspectives and of its causes. The research into the causes encouraged Ibn Lūqā, as well as al-Kindī, to go further than geometrical exposition. They intended explicitly to combine geometry of the vision and physiology of the vision. Thus the status of optics is made clear as is described by Qusṭā ibn Lūqā: 'The best demonstrative science is that in which physical science and geometrical science participate communally, because it takes from physical science the sensory perception and takes from geometrical science demonstrations with the help of lines. I have found nothing where these two disciplines are united in a more beautiful or perfect way than in the science of rays, above all those which are reflected onto mirrors.'[20]

In this way, therefore, for Ibn Lūqā and al-Kindī optics does not reduce further than catoptrics to geometry; on the contrary it is necessary to combine geometry and physics, owing to the properties of visual perception. This course taken by Ibn Lūqā certainly distinguishes him from Euclid, but must not be identified with a new perspective that only emerged later, with the reform by Ibn al-Haytham.

The main object of the book by Ibn Lūqā lies in the study of reflection on plane and spherical convex and concave mirrors, and the diversity of images perceived according to the position of the visible object in relation to the mirror, of its distance from the latter etc. But before undertaking this study, Ibn Lūqā starts by explaining briefly about vision and by recalling some optical results.

His doctrine about vision is of Euclidean and Galenic origin at the same time. Vision takes place by 'a ray which emerges from the eye and falls on visible objects, which will be seen by the ray which falls on them; that on which the visual ray falls is seen by the man and that on which the visual ray does not fall will not be seen by the man.[21] One recognizes quite evidently in Ibn Lūqā's statement the expression of the third definition of *Optics* by Euclid. It remains to make precise the form of this visual ray. Ibn Lūqā writes then:

> The visual ray emerges from the eye in the form of the figure of a cone whose summit is on the side of the seeing eye and whose base is on the side of the visible objects on which it falls. Thus, that onto which the base of the cone falls is perceived by the eye, and that onto which the visual ray does not fall the eye does not perceive. This visual cone extends from the seeing eye in straight lines without curvature. The cone has an angle surrounded by two sides of the cone; this angle is on the side of the visible object, because it is the cause by which a thing is perceived with different magnitude according to whether it gets closer to the eye or further away; it is seen large when close, and small when it is far away.[22]

It is clear that Ibn Lūqā only took up here the ideas contained in the first four definitions of *Optics* by Euclid. But, to this Euclidean material, Ibn Lūqā superposes other elements, Galenic, after which 'this visual ray emerges from the animal soul—*al-rūḥ al-nafsāniyya*, which in Greek is πνεῦμα ψυχικόν—which emerges from the brain to the pupil by way of two open nerves which join the brain to the two eyes, and it emerges from the eye into the air to [reach] visible objects, and it will thus [be] like an organ for man'.[23] This visual ray nevertheless only perceives the visible objects with the help of one or other kind of rays, which, according to Ibn Lūqā, are the solar ray and the fire ray. Each of these two rays 'fills the air with a luminosity without which and outside of which there is no vision'.[24] Ibn Lūqā remains unfortunately silent about the role of the air and of luminosity in vision.

The borrowing by Ibn Lūqā of Galenic elements—brilliantly taken up at the time by Ḥunayn ibn Isḥāq—seems to stem from the powerlessness of the Euclidean doctrine to justify that the visual ray is the instrument of the eye, whilst nevertheless vision is an act of the soul.

If now we come back to catoptric and optical study, we must consider again the concern of Ibn Lūqā to justify and establish that which Euclid could only postulate—but this project is not particular to him; it will appear again, in a still more striking way, with al-Kindī. Thus, after having justified the postulate by Euclid, which states that the same visible object can be perceived in different forms according to the diversity of the angles of the visual ray under which one sees it,[25] he starts his true project, catoptric research. The principal method which he uses in this research is the law of reflection, thus expressed: 'The visual ray, and even all rays, if it meets a polished body, reflects on itself forming equal angles.'[26] In the course of the application of this law, Ibn Lūqā supposes, but without explanation, that the incident ray and the reflected ray lie in the same plane, perpendicular to the plane of the mirror. If it is necessary to identify a fundamental trait of the catoptric research by Ibn Lūqā, it concerns the following: he is interested much more in the angle under which the object is perceived in the mirror than in the image, in an optical sense, of that object.

To illustrate the procedure of Ibn Lūqā, we take the example of proposition 28 of his *Treatise*. He wants to know the reasons why one does not see one's face in certain mirrors. About which mirrors is he talking, and at what distance does this phenomenon take place? In response to this question, Ibn Lūqā mentions the concave mirror, of spherical concavity, when the observer is situated at its centre. The reason for the phenomenon is that 'the visual ray emitted from the eye in this position reflects on itself'.[27]

To prove this proposition, Ibn Lūqā considers such a mirror. Let an arc AB be smaller than a semi-circle whose rotation draws out the surface of a sphere. Let E be the centre of the sphere; the eye is at E (Figure 19.1). Take the visual ray between the two segments AE and EB. It is shown that this ray reflects on itself.

From E to the mirror AB consider as many segments as you like: EC, ED, EG, EH. These segments are equal and each forms two equal angles with the circumference of the circle. Now, writes Ibn Lūqā, 'we have shown that the ray reflects itself on the polished body following equal angles; therefore if we imagine the straight lines EA, EC, ED, EG, EH, EB as rays which meet a polished body which is the mirror, which is on AB, then they meet it at equal angles; each therefore reflects on itself and they are reflected to a single point which is point E, and nothing is seen in the mirror.'[28]

In this proof, Ibn Lūqā has recourse to nothing other than the *Catoptrics* of Pseudo-Euclid, the second and fifth propositions. We also observe that, just as did Pseudo-Euclid in this book, he studies how the object appears in the mirror to the eye of the observer. We finally note that in addition to the propositions of Pseudo-Euclid already mentioned,

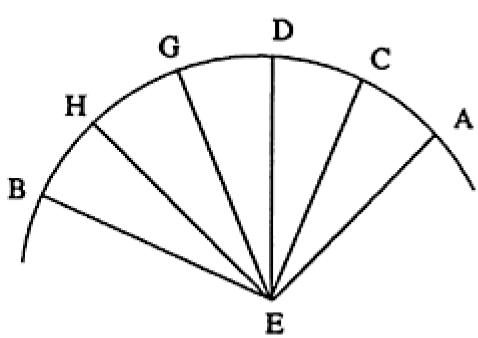

Figure 19.1

Ibn Lūqā makes reference in the course of his study to other propositions from the same book—the seventh, the eleventh, the twelfth; this confirms our affirmation that the Arab authors knew in one way or another a version of this text.[29]

With Ibn Lūqā we are very much in the area of Hellenistic optics and catoptrics. He is known as an eminent translator, and this is therefore an exemplary case. It is following Euclid that he conceives and composes a book where he applies what he has been able to retain of Euclid's *Optics,* but also what he has learnt from one of the versions of *Catoptrics,* as well as perhaps another source, still to be identified, in the tradition of Hero of Alexandria. But for all that his contribution does not reduce to a simple commentary on Euclid or Pseudo-Euclid. In effect he deliberately starts a new research in the domain of entertaining mirrors, amends the Euclidean doctrine on vision and justifies what Euclid had announced as a postulate. The modesty of these results does not manage to conceal the overtly innovatory attitude of Ibn Lūqā, an inclination due not to his own character and not confined to optics: it is a phenomenon of the epoch, and to forget it would prevent an understanding of the works of this time. Did it appear in the research by Ibn Lūqā into burning mirrors? We are ignorant about this for reasons already given. It is this which in any case motivated al-Kindī, a contemporary of Ibn Lūqā, in his philosophical and optical work, in his works on burning mirrors.[30] In these two domains, al-Kindī expressly gives himself the task of exposing the learning of the ancients, of developing what they started and of rectifying the mistakes made. He keeps his word in the three writings on geometrical optics that have reached us. We start by quickly analysing the *Liber de causis diversitatum aspectus [De aspectibus],* and then his book on burning mirrors, before mentioning his other opuscules on physical optics.

In a much more radical way than Ibn Lūqā, al-Kindī wants to prove what Euclid postulated. The first quarter of *De aspectibus* is designed to justify the rectilinear propagation of luminous rays, with the help of geometrical considerations on shadows and on the passage of light through slits, developing thus remarks made in the epilogue added to the *Recension* by Theon of Alexandria of Euclid's *Optics*.[31]

In the first proposition of the book he shows that, if the luminous source and the body illuminated by this source are both spheres of the same diameter *d,* then the shadow is cylindrical and the shadow thrown on a plane perpendicular to the common axis is a circle of the same diameter *d*. Conversely, if the illuminated body and the shadow on a plane have the same diameter *d,* then the luminous source is a sphere of diameter *d.*

In the second proposition, he shows that if the luminous source has a larger diameter than that of the illuminated body, then the shadow is a cone and the shadow thrown on a plane perpendicular to the axis of the cone is a circle of diameter less than that of the illuminated body. He shows next—proposition 3—that if the luminous source has a diameter less than that of the illuminated body, then the shadow is a frustum of the cone and the shadow thrown on a plane perpendicular to the axis of the frustum of the cone is a circle of diameter greater than that of the illuminated body. These three propositions allow al-Kindī to demonstrate rectilinear propagation.

Al-Kindī adds three other propositions designed to establish the same principle definitively. Thus, in proposition 5, he considers a luminous rectilinear source ED (or even a point source D) and an illuminated object AB, itself rectilinear. He affirms that, if the shadow is BG, then the experiment gives BG/BA=EG/DE, from which it follows that D, A and G are aligned (Figure 19.2(a)).

In effect if they were not aligned, then one traces DG, which cuts AB at U (Figure 19.2(b)). The triangles GBU and GED are homothetic and BG/BU=EG/DE. In comparing the two ratios, one would have BU=BA, which is a contradiction.

In the sixth proposition, al-Kindī consider a slit illuminated by a luminous source and establishes rectilinear propagation starting from the image of this slit.

We note that al-Kindī is considering here the rays of luminous sources, which implies that he admits, like so many other authors of antiquity, that these rays and visual rays have identical behaviour with respect to propagation and other laws of optics.

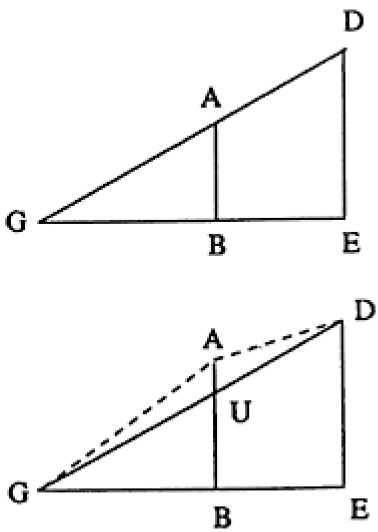

Figure 19.2

Once rectilinear propagation had been established, al-Kindī returned to the theory of vision.[32] He starts by recalling the principal doctrines known since antiquity, to adopt finally that of emission. He justifies his choice by advancing new arguments counter to ancient doctrines, principally against that of the intromission of forms (as among Greek atomists) and that of the emission—intromission of forms (as with Plato). His criticism comes back finally to showing the impossibility of reconciling the doctrine on the intromission of forms, i.e. of non-analysable totalities in their simple elements, and the fact that the perception of an object is a function of its localization in ordinary space. If the doctrine on the intromission of forms was correct, recalls al-Kindī, then a circle in the same plane as the eye would be perceived in all its circularity, which is false. Al-Kindī nevertheless does not accept the Euclidean doctrine on emission without making some serious amendments to it. The visual cone, according to him, and unlike what Euclid believed, is not formed from discrete rays but appears as the volume of continuous radiations.

But in fact the importance of this last amendment—raised already by Ptolemy—lies in the idea on which it is based: that of the ray. After the fashion of Ibn Lūqā, al-Kindī moves away from a purely geometrical concept of a ray: rays are not geometrical straight lines but impressions produced by three-dimensional bodies; or, in the words of al-Kindī

'But a ray is an impression of the luminous body on opaque bodies, of which the name is derived from that of the light as a result of alteration by accidents occurring to the forms which receive this impression. Therefore the impression, with this in which is the impression, all that put together is a ray. But the body which produces this impression is a body which has three dimensions: length, width and depth. Therefore the ray does not follow straight lines between which there would be intervals.'[33]

This criticism of the concept of the ray, in itself important, prepares in some way for a fundamental step which will be taken by Ibn al-Haytham: the separation between light and the straight line along which it propagates. But al-Kindī must again explain the diversity of perception according to the different regions of the cone. He differentiates himself on this occasion from both the position of Euclid and that of Ptolemy by supposing that from every point of the eye a visual cone is emitted.

Thus, after having established rectilinear propagation—to which he returns in the thirteenth proposition to show that it takes place in all directions—and elaborated his doctrine on vision, he returns to the study of mirrors and images from the sixth proposition of his book. It is there that he proves the equality of the angles formed by the incident ray and the reflected ray with the normal to the mirror at the point of incidence. The proof that al-Kindī gives of this law is not only geometrical but also experimental. He places a plane mirror AB and a board UZ parallel to AB (Figure 19.3). He considers a point D on the mirror and traces GD which crosses UZ at H.

On UZ one draws a perpendicular at I and considers a distance IT=HI. One makes a round hole at T. Finally one places another board KL parallel to AB. The experiment by al-Kindī consists then of placing a luminous source on DG or on its extension and of showing that the reflected ray will be along DE.

This 'experimental verification' is inscribed in a long tradition of which one finds traces in the recension by Theon of the *Optics* of Euclid, and which will be rethought in depth by Ibn al-Haytham, as we shall see.

Al-Kindī pursues this same research and considers—proposition 18—a spherical convex or concave mirror to show that, at each point of the sphere, the reflection of a ray will lie on the tangential plane at this point. In proposition 21, al-Kindī examines the position of the virtual image and brings out the idea of symmetry in relation to the mirror. He then studies—proposition 23—the idea of apparent diameter.

The contribution of al-Kindī is not confined to his work on optics and catoptrics. It is as if he had wished to deal with all the problems inherited from ancient optics. Thus he devotes a whole book to burning mirrors. After him, no Arab scholar of renown in the field of optics neglected to include the study of burning mirrors in his research programme. Such at least is the case for the two most important authors: Ibn Sahl and Ibn al-Haytham. This is therefore a central matter in optics and no longer, as was the case in antiquity, a separate special case: moreover, we shall see that this study leads precisely to the inauguration of a new subject area in the tenth century: the anaclastic.

This book by al-Kindī, which has never been correctly analysed up to the present time,[34] is situated, like the other works by the author, both in continuation of the ancient scholars and against them. Al-Kindī intends here to compensate for the shortcomings of the study by Anthemius of Tralles, which he completes. Did not the latter take as an incontestable truth

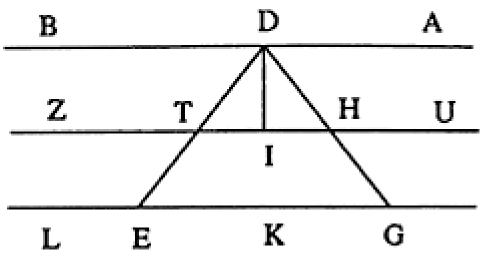

Figure 19.3

the legend according to which Archimedes set alight the Roman fleet, without even proving that it was possible? Did he not work on the construction of a mirror whose twenty-four rays reflected towards a single point, without rigorously determining the distance from this point to the mirror? It is this task that al-Kindī proposes to take up, in fifteen propositions of rather unequal value.

The first four propositions have as their aim the construction of a burning mirror in conical form. It is to this end that he studies in the first three a system formed from two plane mirrors placed on the faces of a dihedron.

The seven following propositions deal with the construction of spherical concave mirrors. The axis of the mirror is always directed towards the sun, and al-Kindī envisages rays falling on the points of the circle which surrounds the mirror; he shows that the corresponding reflected rays meet the axis at the same point. He singles out several cases, according to the ratio of the arc AB—which defines the mirror—to the large circle of the sphere which completes it, considering a spherical concave mirror with axis CD, having the form of a semi-sphere, and to considering on this mirror circles of axis CD (Figure 19.4a).

Let Γ be one of the circles and L its centre; E is the centre of the sphere, R its radius and O the mid-point of EC (Figure 19.4b). We can thus sum up the principal results found by al-Kindī.

1 The solar ray falling on a point A of a circle Γ is reflected towards a point H of the axis CD. The point H remains fixed when A describes Γ.

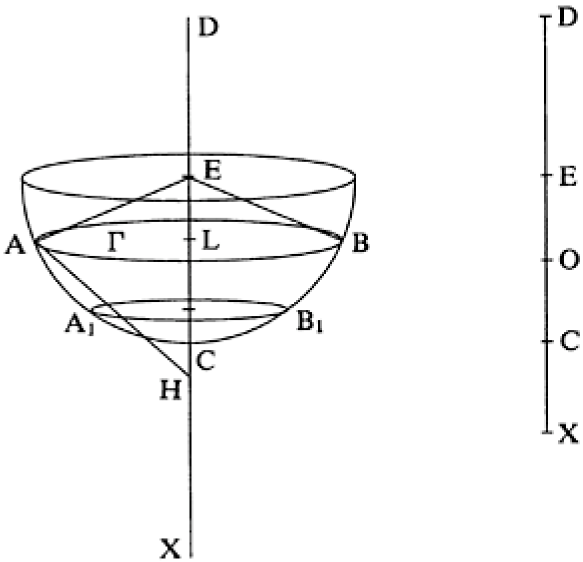

Figure 19.4

2 The point H is dependent on the arc AB which corresponds to this circle Γ; therefore the angle AEB=α.

- H describes OC when $\alpha \in [0, 2\pi/3]$.
- When $\alpha \in]2\pi/3, \pi[$, the point H—towards which is directed the reflected ray—belongs to the half straight line CX.
- The distance from the point H to the centre of the circle Γ is known when the arc AB is known. It is easily established that

$$LH=R \sin(\alpha/2). |\cot an\ \alpha|$$

Thus for a mirror surrounded by an arc AB equal to 2π/3, the reflected rays corresponding to all the solar rays falling on the mirror are concentrated on the segment OC. The rays falling in the vicinity of C are reflected to pass in the vicinity of O. However, if 2π/3<arc AB<π, if one wishes that the reflected rays effectively meet the axis, it is necessary to have a segment of the sphere with centre C.

After study of this mirror, al-Kindī returns to the problem of Anthemius of Tralles: the construction of a system of twenty-five hexagonal mirrors which allow the solar rays falling on their centre to reflect towards one and the same point. He shows that, if the solar rays are parallel to the axis of the central mirror, for thirteen mirrors the problem is simple and leads to a point R. But, for the twelve others, one runs into the difficulty met by Anthemius, and the rays considered reflect towards a different point to the one obtained by the first thirteen, a point situated on the axis of the system near point R.

For the six mirrors surrounding the central mirror, the demonstration by al-Kindī is true; but he affirms without demonstration that the same property applies to the others, which is not altogether exact.

In proposition 14, al-Kindī wants to construct a mirror which is 'much more perfected than that of Anthemius'. Thus, starting with a regular polygon with twenty-four sides, he constructs a regular pyramid of twenty-four faces, so that the solar rays falling on the middle of the bases of these faces taken as mirrors are reflected towards the same point J of the axis of the pyramid. He defines this point J by considering two symmetrical faces in relation to the axis, but does not show that the point J stays the same whichever face one takes. We note that this result is immediate if one takes into account the planes of symmetry of the regular pyramid.

The last part of the *Treatise* by al-Kindī finishes on a text which, once reconstructed, delivers to us the problem of Anthemius: to construct a mirror of a given diameter which reflects the rays towards a given point. The process indicated by him is no other than the construction by points and tangents of a parabola of which one knows the focus and the directrix. Here, the ideas and the method are the same as those of Anthemius, but the proof by al-Kindī is clearer and more ordered, at least if one compares it to what has been transmitted to us in the Greek text by Anthemius, or the Arabic version that we have had the good fortune to discover.

One measures thus all the extensions and importance that al-Kindī has been able to bring to the study of burning mirrors. He examines five mirrors, and thus a much greater number than his Hellenistic predecessors; he refers to the recent translation of Anthemius of Tralles, but so that there is no delay in going further than him. Although he does not adopt from his book the study of the ellipsoidal mirror, it seems that it is because he is only interested in mirrors which can correspond to the legend of Archimedes. The Arabic successors of al-Kindī will very actively follow this study of the propagation of solar rays and of their convergence after reflection, which will profoundly influence the development of the whole of optics, as will be seen.

We also attribute to al-Kindī a small work to show that the 'magnitudes of figures immersed in water are seen as much larger that the figures are more immersed', where he tries, with the aid of reflection, to give an account of the phenomenon of refraction. This treatise by al-Kindī, falsely attributed to a later author, shows that the philosopher did not yet know *Optics* by Ptolemy. We recall finally some short pamphlets where al-Kindī deals in one way or another with the problem of colour. The first is entitled 'Of the carrier body by its nature of colour among the four elements and which is the cause of colour in others'; this body is no other according to al-Kindī than 'the earth'.[35] In the second, he asks himself about 'the cause of the blue colour that one sees in the atmosphere in the direction of the sky and which one believes to be the colour of the sky'.[36] Al-Kindī therefore maintains that this colour is not that of the sky but a mix of the obscurity of the sky and the light of the sun reflected on the particles of the earth in the atmosphere.

IBN SAHL AND THE GEOMETRICAL THEORY OF LENSES

At the turn of the ninth century, a collection of optical writings appeared comprising both translations of Greek books on optics, catoptrics, burning mirrors and physiological optics and new contributions of the Arabic scholars themselves. The ancient bibliographers cite names and titles about which we are barely informed. Thus, in the generation which succeeded that of al-Kindī and Ibn Lūqā, the bibliographer of the tenth century al-Nadīm cites Ibn Masrūr al-Naṣrānī. But, while everything indicates that writing on optics was continuing in this era, very few documents have reached us on geometrical optics; the latter are testimony to an essential preoccupation: the study of burning mirrors.

At present, we have the use of only three writings, of which two belong without doubt to that period—the book by the astronomer ʿUṭārid ibn Muḥammad and the essay of the mathematician Abū al-Wafāʾ al-Būzjānī—and a third which could also be from that period but which we are not sure about: the book by Aḥmad ibn ʿĪsā. Now the writing by ʿUṭārid, as we have shown elsewhere,[37] is a compilation of *Burning Mirrors* by Anthemius of Tralles and of another Greek writing in the tradition of Hero of Alexandria. The commentaries by ʿUṭārid add nothing essential, no more than the book by Aḥmad ibn ʿĪsā. We have shown that it is a compilation of the same sources, to which one must add the *Burning Mirrors* by al-Kindī as well as the essay on figures immersed in water, previously mentioned, which is attributed to him, *Optics* by Euclid, and plenty of other texts. The book by Ibn ʿĪsā is important for knowledge of Greek and Arabic sources in the ninth century. This compilation unites matters which usually belong to independent works. It is thus that one finds in it, besides optics and catoptrics, burning mirrors, the halo and rainbow, and the description of the eye. As regards Abū al-Wafāʾ finally, he applies an elegant method to the construction of the parabolic mirror.

The interest shown in the study of burning mirrors is an essential part of the comprehension of the development of catoptrics, but also of dioptrics, as our recent discovery of a treatise produced between 983 and 985 by the scholar Abū Saʿd al-ʿAlāʾ ibn Sahl testifies. Beginning the study of burning mirrors, Ibn Sahl is the first in history to engage in research on burning lenses: it is the birth of dioptrics. This recent knowledge of the work by Ibn Sahl explains in a new light the work of his successor, Ibn al-Haytham, by precisely locating his historical and mathematical situation.

Before Ibn Sahl, the catoptricians asked themselves about the geometrical properties of mirrors and about the light they produced at a given distance. This is in sum the problem which Diocles, Anthemius and al-Kindī set themselves. Ibn Sahl modifies the question immediately by considering not only mirrors but the instruments burning, i.e. those which are susceptible to light not only by reflection but also by refraction. Ibn Sahl studies then successively, according to the distance of the source (finite or infinite) and the type of lighting (reflection or refraction), the parabolic mirror, the ellipsoidal mirror, the planoconvex lens and the biconvex lens. In each of these sections, he proceeds to a theoretical study of the curve, and then expounds a mechanical process to draw it. For the planoconvex lens, for example, he starts by studying the hyperbola as a conic section and then proceeds to the continuous drawing of an arc of the hyperbola, in order then to take up again a study of the plane tangent to the surface engendered by the rotation of this arc around a fixed straight line, and finally to refind the laws of refraction. To obtain a better understanding of Ibn Sahl's study on lenses we must first explain his views on refraction.

In another essay, which has survived and which has been commentated on by Ibn al-Haytham, written whilst he was examining the fifth book of *Optics* by Ptolemy and entitled 'Proof that the celestial sphere does not have an extreme transparency', Ibn Sahl applies to the study of refraction some concepts already present in the work of Ptolemy. But in this study the notion of the medium holds an important place. Ibn Sahl shows that every medium—including the celestial sphere—is equipped with a certain opaqueness that defines it. But—and this is the true discovery—Ibn Sahl *characterizes* the medium by a certain ratio, which he does in his treatise on *Burning Instruments*. It is precisely this concept of a constant ratio characteristic of the medium which is the masterpiece in his study of refraction in lenses.

At the beginning of this study, Ibn Sahl considers a plane surface GF surrounding a piece of transparent and homogeneous crystal. He next considers the straight line CD along which the light propagates in the crystal, the straight line CE along which it refracts itself in the air, and the normal at G on the surface GF which intersects the straight line CD at H and the ray refracted at E (see Figures 19.5 and 19.6).

Obviously, Ibn Sahl is here applying the known law of Ptolemy according to which the ray CD in the crystal, the ray CE in the air and the normal GE to the plane surface of the crystal are found in the same plane. He writes then, in a brief way, and, according to his habit, with no conceptual commentary:

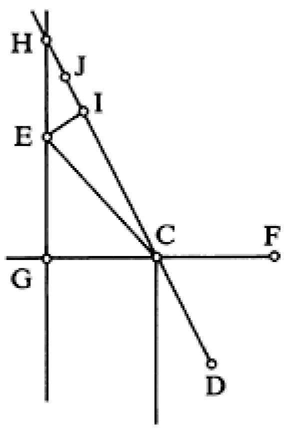

Figure 19.5

> The straight line CE is therefore smaller than CH. We separate from the straight line CH the line CI equal to the line CE; we divide HI into two halves at the point J; we suppose the ratio of the straight line AK to the straight line AB to be equal to the ratio of the line CI to the line CJ; we draw the line BL on the extension of AB and suppose it to be equal to the line BK.[38]

In these few phrases, Ibn Sahl draws the conclusion first that CE/CH<1, which he will use throughout his research into lenses made in this *same* crystal. In effect he does not fail to give this same ratio again, nor to reproduce this same figure, each time that hediscusses refraction in this crystal.

But the ratio is nothing other than the inverse of the index of refraction in this crystal in relation to the air. Considering the i_1 and i_2 as the angles formed respectively by CD and by CE with the normal GH, we have

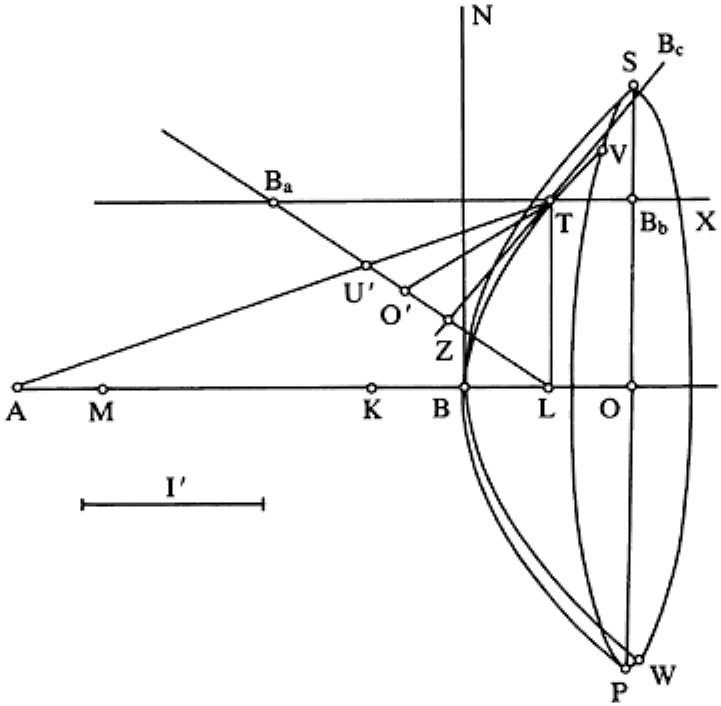

Figure 19.6

$$\frac{1}{n}=\frac{\sin i_1}{\sin i_2}=\frac{CG}{CH}\cdot\frac{CE}{CG}=\frac{CE}{CH}.$$

Ibn Sahl takes on the segment CH a point I such that CI=CE, and a point J at the midpoint of IH. This gives

$$\frac{CI}{CH}=\frac{1}{n}.$$

The division CIJH characterizes this crystal for all refraction.

Ibn Sahl shows, moreover, in the course of his research into the planoconvex lens and the biconvex lens, that the choice of hyperbola to fashion the lens depends on the nature of the crystal, since the eccentricity of the hyperbola is $e=1/n$.

This result allows him to introduce, in the case of refraction, the rule of the inverse return, essential to the study of biconvex lenses.

It is therefore the law of Snellius[39] that Ibn Sahl has found, and which he has effectively conceived. Now this discovery by Ibn Sahl, as well as the application of the law of inverse return in the case of refraction, shows the distance covered since Ptolemy. Thus armed with these conceptual techniques Ibn Sahl takes on the study of lenses.

He shows that solar rays parallel to the axis OB are refracted on the hyperbolic surface, and that the refracted rays converge at A (Figures 19.6 and 19.7).

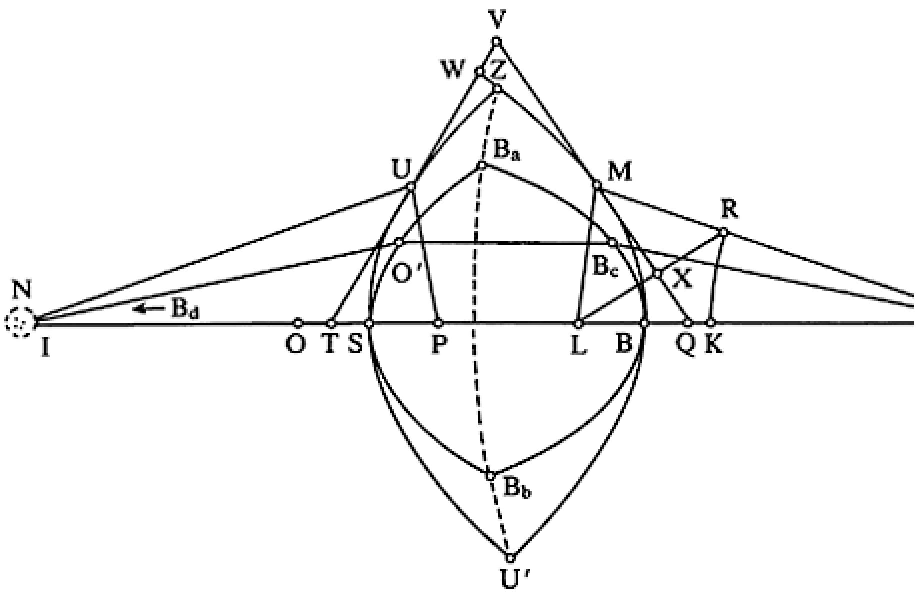

Figure 19.7

He shows next that the luminous rays issuing from the focus N of the hyperboloid and falling on the surface ZSU′ penetrate into this solid, meet ZBU′ and are propagated to the point A; they light up at this point.

Thus Ibn Sahl had conceived and put together an area of research into burning instruments and, perhaps, also dioptrics. But, obliged to think about conical figures other than the parabola and the ellipse—the hyperbola for example—as anaclastic curves, he was quite naturally led to the discovery of the law of Snellius. We understand therefore that dioptrics, when it was developed by Ibn Sahl, only dealt with matters involving the propagation of light, independently of problems of vision. The eye did not have its place within the area of burning instruments, nor did the rest of the subject of vision. It is thus an objective point of view which is deliberately adopted in the analysis of luminous phenomena. Rich in technical material, this new discipline is in fact very poor on physical content: it is evanescent and reduces to a few energy considerations. By way of example, at least in his writings that have reached us, Ibn Sahl never tried to explain why certain rays change direction and are focused when they change medium: it is enough for him to know that a beam of rays parallel to the axis of a planoconvex hyperbolic lens give by refraction a converging beam. As for the question why the focusing produces a blaze, Ibn Sahl is satisfied with a definition of the luminous ray by its action of setting ablaze by postulating, as did his successors elsewhere for much longer, that the heating is proportional to the number of rays.

IBN AL-HAYTHAM AND THE REFORM OF OPTICS

Whilst Ibn Sahl was finishing his treatise on *Burning Instruments* very probably in Baghdad, Ibn al-Haytham was probably beginning his scientific career. It would not be surprising therefore if the young mathematician and physicist had been familiar with the works of his elder, if he cited them and was inspired by them.[40] The presence of Ibn Sahl demolishes straight away the image carved by historians of an isolated Ibn al-Haytham whose predecessors were the Alexandrians and the Byzantines: Euclid, Archimedes, Ptolemy and Anthemius of Tralles. Thus, thanks to this new filiation, the presence of certain themes of research in the writings of Ibn al-Haytham, not only his work on the dioptre, the burning sphere and the spherical lens, is clarified; it authorizes what was not possible previously: to assess the distance covered by a generation of optical research—a distance so much more important, from the historical and the epistemological point of view, now that we are on the eve of one of the first revolutions in optics, if not in physics. Compared with the writings of the Greek and Arab mathematicians who preceded him, the optical work by Ibn al-Haytham presents at first glance two striking features: extension and reform. It will be concluded on a more careful examination that the first trait is the material trace of the second. In fact, no one before Ibn al-Haytham had embraced so many domains in his research, collecting together fairly independent traditions: philosophical, mathematical, medical. The titles of his books serve moreover to illustrate this large spectrum: *The Light of the Moon, The Light of the Stars, The Rainbow and the Halo, Spherical Burning Mirrors, Parabolical Burning Mirrors, The Burning Sphere, The Shape of the Eclipse, The Formation of Shadows, Discourse on Light,* as well as his magnificent book *Optics* translated into Latin in the twelfth century and studied and commented on in Arabic and Latin until the seventeenth century. Ibn al-Haytham therefore started not only the traditional themes of optical research but also others, new ones, to cover finally the following areas: optics, meteorological optics, catoptrics, burning mirrors, dioptrics, the burning sphere, physical optics.

A more meticulous look reveals that, in the majority of these writings, Ibn al-Haytham pursued the realization of a programme to reform the discipline, which brought him clearly to take up each different problem in turn. The founding action of this reform consisted in making clear the distinction, for the first time in the history of optics, between the conditions of propagation of light and the conditions of vision of objects.[41] It led on one hand to providing physical support for the rules of propagation—it concerns a mathematically guaranteed analogy between a mechanical model of the movement of a solid ball thrown against an obstacle, and that of the light[42]—and, on the other hand, to proceeding everywhere geometrically and by observation and experimentation. Optics no longer has the meaning that it assumed formerly: a geometry of perception. It includes henceforth two parts: a theory of vision, with which are also associated a physiology of the eye and a psychology of perception, and a theory of light, to which are linked geometrical optics and physical optics. Without doubt traces of the ancient optics are still detected: the survival of ancient terms, or a tendency, noted by Nazif,[43] to pose the problem in relation to the subject of vision without that being really necessary. But these relics do not have to deceive: their effect is no longer the same, nor is their meaning. The organization of his treatise *Optics* reflects already the new situation. In it are chapters devoted in full to propagation—the third chapter of the first book and Books IV to VII; others deal with vision and related problems. This reform led to, amongst other things, the emergence of new problems, never previously posed, such as the famous problem by Alhazen on catoptrics, the examination of the spherical lens and the spherical dioptre, not only as burning instruments but as optical instruments, in dioptrics; and to experimental control as a practice of investigation as well as the norm for proofs in optics and more generally in physics.

Let us follow now the realization of this reform in *Optics* and in other treatises. *Optics* opens with a rejection and a reformulation. Ibn al-Haytham rejects straightaway all the variants of the doctrine on the visual ray, to ally himself with philosophers who defended an intromissionist doctrine on the forms of visible objects. A fundamental difference remains nevertheless between him and the philosophers, such as his contemporary Avicenna: Ibn al-Haytham did not consider the forms perceived by the eye as 'totalities' which radiate from the visible object under the effect of light, but as reducible to their elements: from every point of the visible object radiates a ray towards the eye. The latter has become without soul, without πνεῦμα ὀπτικόν, a simple optical instrument. The whole problem was then to explain how the eye perceives the visible object with the aid of these rays emitted from every visible point.

After a short introductory chapter, Ibn al-Haytham devotes two successive chapters—the second and the third of his *Optics*—to the foundations of the new structure. In one, he defines the conditions for the possibility of vision, while the other is about the conditions for the possibility of light and its propagation. These conditions, which Ibn al-Haytham presents in the two cases as empirical notions, i.e. as resulting from an ordered observation or a controlled experiment, are effectively constraints on the elaboration of the theory of vision, and in this way on the new style of optics. The conditions for vision detailed by Ibn al-Haytham are six: the visible object must be luminous by itself or illuminated by another; it must be opposite the eye, i.e. one can draw a straight line to the eye from each of its points; the medium that separates it from the eye must be transparent, without being cut into by any opaque obstacle; the visible object must be more opaque than this medium; it must be of a certain volume in relation to the visual sharpness.[44] These are the notions, writes Ibn al-Haytham, 'without which vision cannot take place'. These conditions, one cannot fail to notice, do not refer, as in the ancient optics, to those of light or its propagation. Of these, the most important, established by Ibn al-Haytham, are the following: light exists independently of vision and exterior to it; it moves with great speed and not instantaneously; it loses intensity as it moves away from the source; the light from a luminous source—substantial—and that from an illuminated object—second or accidental—propagate onto bodies which surround them, penetrate transparent media, and light up opaque bodies which in turn emit light; the light propagates from every part of the luminous or illuminated object in straight lines in transparent media and in all directions; these virtual straight lines along which light propagates form with it 'the rays'; these lines can be parallel or cross one another, but the light does not mix in either case; the reflected or refracted light propagates along straight lines in particular directions. As can be noted, none of these notions relate to vision. Ibn al-Haytham completes them with other notions relative to colour. According to him, the colours exist independently from the light in opaque bodies, and as a consequence only light emitted by these bodies—second or accidental light—accompanies the colours which propagate then according to the same principles and laws as the light. As we have explained elsewhere, it is this doctrine on colours which imposed on Ibn al-Haytham concessions to the philosophical tradition, obliging him to keep the language of 'forms', already devoid of content when he only deals with light.

A theory of vision must henceforth answer not only the six conditions of vision, but also the conditions of light and its propagation. Ibn al-Haytham devotes the rest of the first book of his *Optics* and the two following books to the elaboration of this theory, where he takes up again the physiology of the eye and a psychology of perception as an integral part of this new intromissionist theory. This theory will not be embarked on here but will be studied later.

Three books of *Optics*—the fourth to the sixth—deal with catoptrics. This area, as ancient as the discipline itself, amply studied by Ptolemy in his *Optics,* has never been the object of so extensive a study as that by Ibn al-Haytham. Besides the three voluminous books of his *Optics,* Ibn al-Haytham devotes other essays to it which complete them, on the subject of connected problems such as that of burning mirrors. Research into catoptrics by Ibn al-Haytham distinguishes itself, among other traits, by the introduction of physical ideas, both to explain the known ideas and to grasp new phenomena. It is in the course of this study that Ibn al-Haytham poses himself new questions, such as the problem that bears his name.[45]

Let us consider some aspects of this research into catoptrics by Ibn al-Haytham. He restates the law of reflection, and explains it with the help of the mechanical model already mentioned. Then he studies this law for different mirrors: plane, spherical, cylindrical and conical. In each case, he applies himself above all to the determination of the tangent plane to the surface of the mirror at the point of incidence, in order to determine the plane perpendicular to this last plane, which includes the incident ray, the reflected ray and the normal at the point of incidence. Here as in his other studies, to prove these results experimentally, he conceives of and builds an apparatus inspired by the one that Ptolemy constructed to study reflection, but more complicated[46] and adaptable to every case. Ibn al-Haytham also studies the image of an object and its position in the different mirrors. He applies himself to a whole class of problems: the determination of the incidence of a given reflection in the different mirrors, and conversely. He also poses for the different mirrors the problem with which his name, is associated: given any two points in front of a mirror, how does one determine on the surface of the mirror a point such that the straight line which joins this point to one of the two given points is the incident ray, whilst the straight line that joins this point to the other given point is the reflected ray. This problem, which rapidly becomes more complicated, has been solved by Ibn al-Haytham.[47]

Ibn al-Haytham pursues this catoptric research in other essays, some of which are later than *Optics,* such as *Spherical Burning Mirrors*.[48] It is in this essay of a particular interest that Ibn al-Haytham discovers the longitudinal spherical aberration; it is also in this text that he proves the following proposition:

On a sphere of centre E let there be a zone surrounded by two circles of axis EB; let IJ be the generator arc of this zone, and D its mid-point. Ibn al-Haytham has shown in two previous propositions that to each of the two circles is associated a point of the axis towards which the incident rays parallel to the axis reflect on this circle. He shows here that all the rays reflected on the zone meet the segment thus defined: if GD is the medium ray of the zone, the point H is associated with D, and the segment is on either side of H. The length of this segment depends on the arc IJ (Figure 19.8).

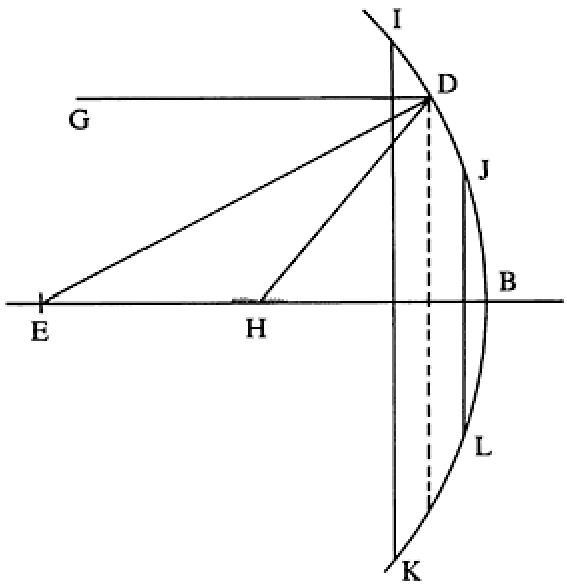

Figure 19.8

The seventh and last book of *Optics* by Ibn al-Haytham is devoted to dioptrics. In the same way as he did for catoptrics, Ibn al-Haytham inserts in this book the elements of a physical—mechanical—explanation of refraction. Moreover, his book is completed by his essays, such as his treatise on the *Burning Sphere* or his *Discourse on Light,* where he comes back to the notion about the medium, following Ibn Sahl.

In this seventh book of *Optics,* Ibn al-Haytham starts by taking on the two qualitative laws of refraction, and several quantitative rules, all controlled experimentally with the help of an apparatus that he conceives of and builds as in the previous case. The two quantitative laws known by his predecessors, Ptolemy and Ibn Sahl, can be expressed thus: (1) the incident ray, the normal at the point of refraction and the refracted ray are in the same plane; the refracted ray approaches (or moves away from) the normal if the light passes from a less (respectively more) refractive medium to a more (respectively less) refractive medium; (2) the principle of the inverse return.

But, instead of following the way opened up by Ibn Sahl through his discovery of the law of Snellius, Ibn al-Haytham returns to the ratios of angles and establishes his quantitative rules.

1 The angles of deviation vary in direct proportion to the angles of incidence: if in medium n_1 one takes $i'>i$, one will have, in medium n_2, $d'>d$ (i is the angle of incidence, r the angle of refraction and d the angle of deviation; $d=|i-r|$).
2 If the angle of incidence increases by a certain amount, the angle of deviation increases by a smaller quantity: if $i'>i$, $d'>d$, one will have $d'-d<i'-i$.
3 The angle of refraction increases in proportion to the angle of incidence: if $i'>i$, one will have $r'>r$.
4 If the light penetrates from a less refractive medium into a more refractive medium, $n_1<n_2$, one has $d<\frac{1}{2}i$; in the opposite path, one has $d<(i+d)/2$, and one will have $2i>r$.
5 Ibn al-Haytham takes up again the rules stated by Ibn Sahl in his pamphlet on *The Celestial Sphere;* he affirms that, if the light penetrates from a medium n_1, with the same angle of incidence, into two different media n_2 and n_3, then the angle of deviation is different for each of these media because of the difference in opaqueness. If for example n_3 is more opaque than n_2, then the angle of deviation will be larger in n_3 than in n_2. Conversely, if n_1 is more opaque than n_2, and n_2 than n_3, the angle of deviation will be larger in n_3 than in n_2.

Contrary to what Ibn al-Haytham believed, these quantitative rules are not all valid in a general sense.[49] But all are provable within the limits of the experimental conditions effectively envisaged by Ibn al-Haytham in his *Optics:* the media are air, water and glass, with angles of incidence which do not go above 80°.

Ibn al-Haytham devotes a substantial part of the seventh book to the study of the image of an object by refraction, notably if the surface of separation of the two media is either plane or spherical. It is in the course of this study that he settles on the spherical dioptre and the spherical lens, following thus in some way the research by Ibn Sahl but modifying it considerably; this study of the dioptre and the lens appears in effect in the chapter devoted to the problem of the image, and is not separated from the problem of vision. For the dioptre, Ibn al-Haytham considers two cases, depending on whether the source—punctual and at a finite distance—is found on the concave or convex side of the spherical surface of the dioptre.[50]

Ibn al-Haytham next studies the spherical lens, giving particular attention to the image that it gives of an object. He restricts himself nevertheless to the examination of only one case, when the eye and the object are on the same diameter. Put another way, he studies the image through a spherical lens of an object placed in a particular position on the diameter passing through the eye. His procedure is not without similarities to that of Ibn Sahl when he studied the biconvex hyperbolic lens. Ibn al-Haytham considers two dioptres separately, and applies the results obtained previously. It is in the course of this study of the spherical lens that Ibn al-Haytham returns to the spherical aberration of a point at a finite distance in the case of the dioptre, in order to study the image of a segment which is a portion of the segment defined by the spherical aberration.

In his treatise on the *Burning Sphere,* one of the peaks of research in classical optics, Ibn al-Haytham explains and refines certain results on the spherical lens which he had already obtained in *Optics*. However, he returns to the question of the burning by means of that lens. It is in this treatise that we encounter the first deliberate study of spherical aberration for parallel rays falling on a glass sphere and undergoing two refractions. In the course of this study, Ibn al-Haytham uses numerical data given in the *Optics* by Ptolemy for the two angles of incidence 40° and 50°, and, to explain this phenomenon of focusing of light propagated along trajectories parallel to the diameter of the sphere, he returns to angular values instead of applying what is called the law of Snellius.

In this treatise on the *Burning Sphere,* as in the seventh book of his *Optics* or in other writings on dioptrics, Ibn al-Haytham exposes his research in a somewhat paradoxical way: while he takes a lot of care to invent, fashion and describe some experimental devices that are advanced for this age, allowing the determination of numerical values, in most cases he avoids giving these values. When he does give them, as in the treatise on the *Burning Sphere,* it is with economy and circumspection. For this attitude, already noted, at least two reasons can perhaps be found. The first is in the style of the scientific practice itself: quantitative description does not yet seem to be a compelling norm. The second is no doubt linked; the experimental devices can only give approximate values. It is for this reason that Ibn al-Haytham took into account the values which he had borrowed from the *Optics* by Ptolemy.

KAMĀL AL-DĪN AL-FĀRISĪ AND THE DEVELOPMENT OF QUANTITATIVE RESEARCH

With Ibn Sahl and Ibn al-Haytham, we have just followed a half-century of research inscribed forever in the history of optics. What impact did the works of these two mathematicians have on their Arab successors? What was the effect of the reform by Ibn al-Haytham, in particular on the future Arabic research into optics? The state of our knowledge does not permit us to give a satisfactory response to these questions. We are able simply to show that the book *Burning Instruments* by Ibn Sahl has been copied by al-Ghundijānī, who was working on astronomy and on optics in the second half of the eleventh century and the beginning of the twelfth, and who commented on other works, such as those by Abū al-Wafā' al-Būzjānī on the burning parabolic mirror. In the middle of the twelfth century, a judge in Baghdad, Ibn al-Murakhkhīm, who busied himself with optics, copied the book by Ibn Sahl as well as his essay on the transparency of the celestial sphere, going exactly from the copy by Ibn al-Haytham.[51] If we mention these traces, it is in order to suggest how it would be adventurous to conclude that the writings of Ibn Sahl—whose treatise was only discovered less than ten years ago—and those of Ibn al-Haytham were not known to their successors. It is no less true that, amongst these, certain of them who composed works designed for elementary teaching and not for research, such as Naṣīr al-Dīn al-Ṭūsī (died in 1274), have continued to comment on Euclid.

The first contribution that reached us in the tradition of Ibn al-Haytham is that by the Persian Kamāl al-Dīn al-Fārisī, born in 1267 and died on 12 January 1319. Al-Fārisī wrote a 'Revision' of *Optics* by Ibn al-Haytham,[52] i.e. an explanatory commentary, sometimes critical. He did the same thing with other treatises by the same scholar, notably *The Burning Sphere* and *The Rainbow*. In all these writings, al-Fārisī is pursuing the realization of the reform by Ibn al-Haytham, sometimes even counter to the latter, and succeeds there where his predecessor had failed: this is the case for the explanation of the rainbow. To this important success—it is the first correct explanation of the form of the rainbow—he adds an advance in the understanding of the phenomenon of colours. In addition, he takes up the quantitative research initiated by Ibn al-Haytham, to give it a new extension and to bring to a close the project of his predecessor.

In his commentary on *The Burning Sphere* by Ibn al-Haytham, al-Fārisī gives a quantitative study which for a very long time remained the most developed. Al-Fārisī researched an algorithm which on the one hand could express the functional dependence between the angles of incidence and the angles of deviation, in order to deduce the values of the deviation for any incidence, for two defined media, and, on the other hand, starting from a small number of measurements—two—to interpolate for all the degrees in the interval. The procedure by al-Fārisī is the following: he shares the interval [0°, 90°] into two subintervals where he approaches $f(i)=d/i$ by an affine function defined on [40°, 90°] and by a polynomial function of the second degree on the interval [0°, 40°]. He next links the two interpolations, by setting the first difference to be the same at the point $i=40°$, or, in other words, by making curves tangential at this point. This method was borrowed by al-Fārisī from astronomers.[53]

Following his commentary on *The Burning Sphere,* al-Fārisī takes up again the explanation of the rainbow. To introduce experimental norms, where Ibn al-Haytham had failed, al-Fārisī abandons a direct and complete study of the phenomenon in order deliberately to apply the method of models: the glass sphere filled with water will function as the droplet of water in the atmosphere. The mathematically guaranteed analogy allowed al-Fārisī to start with the study of two refractions, with between them one or two reflections in the interior of the sphere, to explain the shape of the principal arc and secondary arc, as well as the reversed order of colours in each of these two arcs.[54]

In his explanation of the colours of the two arcs, al-Fārisī is led to modify the doctrine of Ibn al-Haytham, at least in places. In the course of an experiment in the dark-room, al-Fārisī was able to notice that the production and the multiplicity of colours are a function of both the positions of the images and their luminous intensity. The colours of the arc are for him a function of the combination of reflection and refraction, or, in his own words: ‘the colours of the arc are various, close, between the blue, the green, the yellow and the blackish red, and are the result of the image from a strong luminous source reaching the eye by reflection and refraction, or a composition of the two’.[55] One is therefore far from Ibn al-Haytham: the colours cease to exist independently from the light in their opaque bodies.

Such, briefly outlined, are the new directions in research engaged upon by Kamāl al-Dīn al-Fārisī. To these accepted facts, it is necessary to add a mass of results and significant views throughout his ‘revisions and commentaries’ of the optical works of Ibn al-Haytham. The spread of his voluminous book, where he comments on and revises *Optics* by Ibn al-Haytham, testifying to the number of manuscripts, their date and the place where they are found, as well as the diffusion of another writing where he again takes up the principal themes without proof,[56] do not seem consequently to have relegated to the shadow the book by Ibn al-Haytham, but let it be seen that the learning of optics did not stop after the compiling of the work by al-Fārisī about 1300. But the only substantial study after that by al-Fārisī that we know in this field remains the book by the astronomer Taqī al-Dīn ibn Maʿrūf, the compiling of which was finished in 982/1574.[57] But Ibn Maʿrūf only summarizes the book by al-Fārisī, without providing any contribution of his own. Nevertheless, at almost exactly the same time, the posterity of the book by Ibn al-Haytham was guaranteed in other climes, and in languages other than Arabic, in Europe and notably in Latin.

NOTES

1 It concerns for example the writing by Jibrāʾīl ibn Bakhtīshūʿ (d. 828) on the eye, which has not reached us; or that by Ibn Māsawayh *(The Alteration of the Eye, Daghal al-ʿAyn)* which has been preserved.

2 On the Arabic translation of Anthemius of Tralles, see Rashed (1992).

3 Al-Fārābī, *Iḥṣāʾ al-ʿUlūm*, pp. 98–102.

4 The study of the works of Qusṭā ibn Lūqā and Abū Isḥāq al-Kindī, both in the ninth century, shows that they were familiar with the *Optics* by Euclid, one or other version of *Catoptrics* by Pseudo-Euclid. As regards the *Optics* attributed to Ptolemy, we still do not know very well when it was translated. The first true evidence of the existence of this translation is that of Ibn Sahl, relatively late—in the last quarter of the tenth century. See Rashed (1993a).

5 On the works on burning mirrors, see *Dioclès, Anthémius de Tralles, Didyme et al., Sur les Miroirs Ardents,* edited and translated by Rashed (forthcoming).

6 The Arabic translation of Ibn al-Biṭrīq has been edited by A.Badawi; see Aristotle, *Meteorologics*.

7 Badawi (1968); see the Olympiodorus text, 144 *et seq.;* see also the text by Alexander of Aphrodisias, 26 *et seq*.

8 Cf. Ḥunayn ibn Isḥāq (edited and translated by M.Meyerhof); and P.Sbath and M.Meyerhof (1938).

9 Helmut Gätje (1967).

10 See the correspondence by Qusṭā ibn Lūqā published by Khalil Samir, p. 156.

11 Thus the book by Ibn Lūqā on *The Reasons for What is Produced in Mirrors as Regards the Diversity of Perspectives* was compiled for the Abbasid Prince Aḥmad, son of Caliph al-Muʿtaṣim, who ruled from 833 to 842.

12 Cf. the introduction of the writing attributed to Diocles, note 5.

13 About the book entitled *Kitāb ʿilal mā yaʿriḍu fī al-marāyā al-muḥriqa min ikhtilāf al-manāẓir.*

14 See al-Nadīm, pp. 317, 318, 320.

15 Compare the titles given by al-Nadīm.

16 Al-Kindī, 'Al-Kindī, Tideus and Pseudo-Euclid'. (The recently-discovered critical commentary on Euclid's *Optics* is forthcoming.)

17 *Kitāb al-Shuʿāʿāt*, MS Khuda-Bakhsh Library 2048.

18 Qusṭā ibn Lūqā, *Kitāb fī ʿilal mā yaʿriḍu fī al-marāyā min ikhtilāf al-manāẓir*, MS 392, Astan Quds, Meshhed.

19 It is the title *'ilm al-manāẓir* that Thābit ibn Qurra retains in his *al-Risāla al-mushawwiqa ilā al-'ulūm*, MS 6188, Malik, Tehran.
20 *Op. cit.,* fol. 2r.
21 *Ibid.,* fol. 3v–4r.
22 *Ibid.,* fol. 4r.
23 *Ibid.*
24 *Ibid.*
25 *Ibid.,* fol. 4r–4v.
26 *Ibid.,* fol. 6r.
27 *Ibid.,* fol. 13r.
28 *Ibid.,* fol. 13v.
29 Ibn Lūqā uses proposition 7 of the *Catoptrics* of Pseudo-Euclid in his proposition 22, and the propositions 11 and 12 in proposition 30.
30 Cf. the article 'Al-Kindī', *Dictionary of Scientific Biography,* vol. XV, 1978, pp. 261–6.
31 On the influence of Theon on al-Kindī, see the commentaries by Alex Björnbo, in al-Kindī,'Al-Kindī, Tideus and Pseudo-Euclid'.
32 See Lindberg (1971b).
33 Al-Kindī, 'Al-Kindī, Tideus and Pseudo-Euclid', *Liber de causis…,* proposition 11; cf. also *L'oeuvre optique d'al-Kindī,* edited by R.Rashed (Leiden).
34 Cf. note 17. We give a critical edition and a French translation of this text (see previous note).
35 Al-Kindī, *Rasā'il al-Kindī*, vol. 2, pp. 64–8.
36 *Ibid.,* pp. 103–8.
37 Cf. note 5.
38 Ibn Sahl, *Les instruments ardents,* in Rashed (1993a:24).
39 *Ibid.,* pp. xxix–xxxiv; and Rashed (1990).
40 Rashed (1991d), especially p. lxxiii.
41 Rashed (1970a; 1978b).
42 Rashed (1970a:281 et seq.).
43 Nazif (1942–3:763) for example.
44 Ibn al-Haytham, *Kitāb al-Manāẓir*, p. 189.
45 About the famous 'problem of Ibn al-Haytham', brilliantly analysed by Nazif (1942–3:487–521).
46 *Ibid.,* pp. 685–90.
47 About the problem of Ibn al-Haytham; see note 45.
48 *Al-marāyā al-muḥriqa bi-al-dā'ira, bi-al-dā'ira*, fourth treatise in Ibn al-Haytham, *Majmū' al-rasā'il*. See also Wiedemann (1909) and Winter and Arafat (1950).
49 Nazif (1942–3:720–3) and Rashed (1968:201–4).
50 Rashed (1993a: ch. 2).
51 *Ibid.,* pp. cxxxix–cxlii.
52 Kamāl al-Dīn al-Fārisī, *Tanqīḥ al-manāẓir*.
53 Rashed (1993a:lx–lxviii).
54 Rashed (1970b).

55 al-Fārisī, *Tanqīḥ al-manāẓir*, vol. II, p. 337.

56 About his writing *al-Baṣā'ir fī 'ilm al-manāẓir*, MS Istanbul, East Efendi 2006, Suleymaniye.

57 *Kitāb nūr ḥadaqat al-abṣār wa nūr ḥadīqat al-anẓār*, MS Oxford, Bodleian Library, Marsh 119.

20
The emergence of physiological optics

GÜL A.RUSSELL

'There is more to seeing than meets the eyeball'

N.R.Hanson

INTRODUCTION

In the history of physiological optics, the correct identification of the site of the projections from the retina in the striate cortex by Munk (1839–1912) marked the end of an era. Subsequently the task changed from searching for the location of perception to determining the nature of its central mechanisms, or from 'where' in the brain to 'what' occurred in the visual cortex that enabled us to perceive the world.[1]

Intellectually, the concept of point to point organization of visual centres in the cerebral cortex had its historical antecedents. Descartes (1596–1650) was credited with a punctate remapping of the retinal image along the central pathways. Underlying his belief that the optic tract projected to the pineal gland, was the notion of central re-projection.[2] Prior to Descartes, Kepler (1571–1630) had established that an inverted image was formed in the eye by means of the lens focusing the rays of light from each point on the surface of the object to a corresponding point on the retina. Diverging from previous theories, based on the work of such anatomists as Felix Platter (1536–1614), Kepler shifted the emphasis from the lens onto the retina as the light-sensitive surface in the eye. He dissociated the analysis of its optical mechanism from the problematic issue of how an inverted retinal image could be reconciled with a veridical perception of the world.[3]

Historically the very formulation of the notion of a projected image is of crucial significance. It provided a radical solution to the persistent ancient problem of how the external world was perceived through the sense of sight. By bringing together the physics of light and the anatomy of the eye, such a concept marked the beginning of physiological optics. The background to its emergence in the Islamic civilization with Ibn al-Haytham (Latin: Alhazen, *c*. 965–*c*. 1040) will be considered in terms of the following categories: (a) pre-optical theories of vision which were inherited as part of the Graeco-Hellenistic scientific legacy; (b) through criticisms of these theories, the appearance of new elements; (c) divergence from the traditional approach with a corresponding point theory of the ocular image and a synthesis of optics and anatomy.[4]

PRE-OPTICAL THEORIES OF VISION

The Greek approach to vision was influenced by the sense of touch where sensory acquaintance was entirely dependent upon physical contact between the object and the observer's bodily surface. To tactually 'feel' something meant mechanical contact with differing types of surface which determined our sense of wetness, roughness, or smoothness. Once such contact was established between the object and the skin, sensory perception (tactile awareness) was both immediate and complete.[5]

By comparison, the means of contact between the eye of the observer and the object was unclear. The essential question for the Greeks was to determine how the eye could establish remote contact with the object in view of the absence of any apparent physical continuity. The obvious conclusion was that vision occurred using an indirect method of contact with the object via an intermediary agent.

The Greek theories emerged therefore as a series of attempts to discover in analogy with the tactile sense, the means of contact between the eye of the observer and the visible object. The logical possibilities that they explored involved the mediation of (a) a replica cast off from the object to the eye; and (b) an invisible visual power or ray projected from the eye to the object. As with touch, visual perception was the immediate result of either form of contact.[6]

Object-copy or *Eidola* theory

In the view that was developed by the Atomists and specifically by Epicurus (*c*. 341–270 BC), objects continually shed their replicas in all directions. These travelled in a straight line through the air as coherent assemblies or convoys of atoms, retaining the orientation, shape and colour that they had on the surface of the object from which they originated. These thin films (called *eidola*) entered the observer's eye. The visual sensation or awareness was due to this indirect contact with successive *eidola* which conveyed all of the visible qualities of an object.[7]

Emission theories: The blind man's stick

Visual ray

The alternative conceptual position to object copy was the belief that the eye emitted invisible rays which made contact with the object, giving rise to visual awareness. The rays were axiomatically believed to travel only in straight lines, radiating in a geometrical cone of vision, which extended infinitely from the eye. The apex of the cone was in the eye, and with increasing distance of the line of view, there was a corresponding increase in the area of the base of the cone. In other words, the longer the distance travelled by visual rays, the greater the size of the visual field. Vision occurred when an object was encountered by rays within the cone.[8]

The visual ray then was the indirect means of putting the eye in touch with objects of sight. An analogy that was implicit, even when it was not explicitly made was the blind man using a stick, as a tactile extension, with which to feel objects beyond the reach of his hand.[9] A more exact analogy would be to conceive of the blind man holding a bundle of sticks, radiating forward like the ribs of an umbrella.

The geometry of Euclid (fl. 300 BC) had given this position a considerable degree of credibility. It was further refined particularly by the experimental optics of Ptolemy (fl. 127–148) where the Euclidean cone of discrete geometrical lines acquired physical reality as a continuous bundle of radiation.[10] In combining a theoretical notion of tactile-visual ray with the rigorous deductive system of geometry, this theory could both identify and satisfy visual problems which were otherwise inexplicable. For example, using the visual angle at the apex of the cone, it could explain the perception of size in relation to distance of objects and avoid the atomist's dilemma concerning the sight of a mountain. (Even if the form of a large object were conceived to shrink sufficiently to pass through the small opening of the eye, how could it then retain information with regard to its original size?) The small size of the visual angle, however, indicated the great distance over which the mountain was sensed.[11]

Furthermore, since invisible tendrils were assumed to travel from the eye to the visible object in straight lines, just as the flight of an arrow, their mode of propagation was described with reference to the laws of deflection, using mechanical analogies and the science of mirrors (catoptrics).[12] The visual rays were considered to rebound from polished surfaces, that is, surfaces which were compact with no pores for rays to go through, in the same way an arrow was deflected from a bronze shield. This provided the basis for the explanation of how objects were seen by reflection due to mirrors. The operative principle was the equality of the angles of incidence and deflection or rebound.[13] For example, looking into a mirror, held at an acute angle to the direction of gaze, we see the objects to the side, whereas in a mirror held at right angles to us, we see ourselves. It was explained on the basis of the deflection of the tactile-visual ray in the mirror. The angle of rebound being equal to the angle of incidence, it made contact with objects to the side of the observer. It was as if the blind man's stick was bent at right angles without his awareness of the bend. Facing the mirror straight on, the visual ray was bounced back and touched the observer's own face. Here the blind man's stick was folded upon itself. In spite of its impressive capacity to deal with such questions of reflection, as well as size and distance, the theory still had severe limitations. Since the visual rays must inevitably weaken with distance, the question of how they could encompass the whole sky to see the stars remained one of its major problems.[14]

Variations on the visual ray: Plato and the Stoics

In the earlier theory of Plato (*c*. 427–347) the emission from the eyes, which was conceived of as internal fire, is combined with the surrounding external light to form an intermediary between the eye and the object of sight. Vision occurred when this coalescence of visual 'fire' and daylight, a homogeneous single body, made contact with an emanation from the object.[15] What constituted the blind man's stick in Plato was the fusion of visual light and daylight. Furthermore, the visual contact was not between the stick and the object itself, but between the stick and the emanation from the object, which was not an *eidolon* but colour.[16] His position gains conceptual power in providing an explanation of why vision occurred only in the presence of light, despite the tactile nature of the contact between the eye and the object. It could also successfully account for perception of objects at a great distance without recourse to the implausible notion of infinitely extensible rays.

The Stoics initiated a physiological substrate into tactile theories with their notion of *pneuma*. Originally conceived of as a mixture of air and fire, the *pneuma* became associated with the body humours. In the presence of light, a specific *pneuma* stressed the column of air between the eye and the object, causing it to become stretched taut like a rod. The unilluminated air was regarded as too slack to be tensioned by the *pneuma* or to respond to pressure. In this fashion, the *pneuma*-tensioned air formed a cone with its apex in the eye. Visible objects encountered within the base of the cone would be felt and reported back by a rod, as it were, of compressed air similar to the way the blind man uses a stick to feel objects beyond the reach of his hand.[17] The Stoics also compared tactile mediated vision to the impact from an 'electric' eel, transmitted through the net, rod, and hands to the fisherman.[18]

In these theories, the presence of light enables tactile contact or link to be established between the eye and the object. Without light, the visual power, (whether it is the ray or the *pneuma*), cannot tension the air. Thus in the dark, contact is not possible because the air ceases to serve as a tactile stick with which to reach the object. To continue the analogy, it is as if the blind man loses the rigidity of his stick.

The Galenic synthesis

In Galen (*c*. 129–199/200), we see for the first time an uncompromisingly medical approach to vision. His eclectic theory, brought into the established geometry of the perspective cone, a strong emphasis on ocular anatomy.[19] The Stoic theory of the *pneuma* as an essential agent for vision provided for Galen a perfect means to interpret his detailed knowledge of the eye. Originating in the ventricles of the brain, it was in constant flow to the eyes via the optic nerves, which were hollow. There it filled the crystalline body which Galen regarded as the principal organ of sight. This was reinforced by Galen's knowledge of the effect of cataracts which were believed to occur between the crystalline body and the cornea, obstructing vision. As their removal restored sight, it was thought that they blocked the flow of the *pneuma* from the crystalline humour via the pupil to the surrounding air.[20]

In Galen's theory the *pneuma* did not need to emerge far from the eye because upon its immediate contact, the air was instantanously altered (in the presence of light) to become a direct, sensory extension of the organ of vision. Considered in geometrical terms, a cone of sensitivity was formed with visual lines running from its apex in the pupil to the objects of sight in the distance. For Galen, the pneumatized air is not a substitute for the blind man's stick; it becomes a substitute for the blind man's arm itself, analogous to a phantom limb.[21] Perception occurred where the base of the cone encountered the visible object. However, Galen also presented the view that impressions were returned to the crystalline humour as the principal organ of vision and then conveyed via the retina and the 'hollow' optic nerve to the brain as the ultimate seat of sensation and perception.[22]

Transition away from tactile theories

In Aristotle (383–322 BC) we have an approach which is not obviously tactile. The eye does not contact visible objects by its own activity, that is by an emission of a tactile ray or *pneuma*. Nor does it receive filmy copies or 'eidola' of the object. Like all sensation, vision is a passive process.[23] What the sense organs receive is the form of the sensory object without its matter, just as the wax receives the impression of the signet ring without its metal. Each sense organ, however, is affected by impressions from the object appropriate or specific to it. It is in the experience of perceiving that the eye from being potentially capable of sight becomes an actual sensory organ.[24]

Aristotle only goes so far as to identify, in his description of the senses, the necessary conditions for visual experience. First of all, he specifies the essential property of the object of sight as colour, including brightness and darkness, by means of which visible qualities can be perceived. He further emphasizes transparency as a prerequisite for its transfer to the eye. Thus for vision to occur, the coloured object must be separated from the eyes by a transparent medium, and what causes transparency in the medium is light. In Aristotle, light is neither a material substance, nor a movement. It is the state of transparency for the medium (air) through which colours can actually be seen at a distance. It is also by virtue of their transparency that the eyes (or the 'eye jelly') can simultaneously take on colours. In analogy with the signet ring, a green object colours the eye green.[25] There is no explanation as to how this takes place, nor what happens within the eye.[26]

Aristotelian ideas subsequently formed the nucleus of arguments against the tactile approach to vision. Although the notion of emission from the eye itself was vulnerable to attack, the impressive achievements of the emission theories in dealing with the problems of reflection and the perception of distance, size and position were not. Therefore, in the commentaries on Aristotle, there were attempts at an eclectic approach which utilized the geometrical principles and the very mechanics of the visual ray to defend and in the process to revise his hypotheses.[27] Commentators such as Alexander of Aphrodisias in the third century, argued that nothing is emitted from the eye to the object. At the same time, however, in considering the transmission of the visible qualities (colours) by the intervening transparent medium, he made use of the visual cone and the principle of rectilinear propagation of the tactile theories. Objects were seen through a cone along straight lines. Although the perception of their size was determined by the visual angle, placed at the eye, the cone itself was determined by the object of sight at its base, and not by any emission from the eye.[28]

John Philoponus in the sixth century was unambiguous in his view that if visual rays were emitted in straight lines and deflected from smooth surfaces according to the law of equal angles, then the action *(energeia)* of coloured and luminous objects on the eye could also be assumed to occur in straight lines and be reflected from mirrors according to the law of equal angles. In fact, replacing the visual ray with Aristotle's hypothesis had the advantage of avoiding the absurdities of emission and saving the phenomena. Here, in treating light and colour in a parallel fashion, Philoponus had gone beyond Aristotle. He had modified the concept of light from a change of state to qualitative 'movement' (i.e. leap) which took place all at once, like Aristotle's action of colour on the eye.[29]

Thus in late antiquity, a new emphasis emerged in response to Aristotelian ideas. Its eclecticism also embodied the influence of the Neoplatonic principle of emanation (visibly exemplified by the radiation of light from the sun), and the more precise atomist ideas of space and motion.[30] Vision was due to a qualitative movement (or a 'discontinuous leap') of light from objects of sight, which conveyed (via colour) their visible qualities to the eye. And what is more, this transmission could sustain geometrical analysis.[31]

The mechanics of vision in Greek theories

The Greek explanations of vision can be reduced to two main types of theory: (a) the 'object-copy' theory where the eye receives a replica of the object, an *eidolon,* and (b) the more comprehensive and successful 'tactile theory' where a power from the eye as a cone of radiation extends to the visible objects. The non-tactile approach initiated by Aristotle does not constitute a comparable theory although it may subsequently have served as the source of attempts to discredit both.

Despite their apparent differences, the Greek theories of vision were predicated on shared assumptions. First of all, sensory awareness was regarded as a veridical record of reality. What was communicated to the eye and thereby to the mind was a qualitative copy of the external world. This concept was empirically supported by the actual appearance of one's face in the pupil of another person as in a mirror.[32] Hence, the objects of visual sensation were coherent entities. They were perceived holistically, whether by means of a material copy (an *eidolon*), the felt impression, representation or form of the sensory object.[33]

The concept of 'copy' meant that sensory experience was the only means of construing visual theory and the only model to explain perception. At the same time, it was clearly recognized that the senses were not infallible and that there could be a discrepancy between the characteristics of objects and our perceptions of them. The moon, the sun and the stars were seen as though at the same distance when their relative distances from the viewer were vastly different. Attempts to come to terms with this problem is exemplified by the 'moon illusion'[34]: the moon appeared larger at the horizon than when it was overhead even though its physical size was the same in both positions.[35] This insight found a practical application in scenography (scene-painting) and architecture where columns were made slightly divergent or non-parallel to appear parallel to the observer. In fact, optics was a branch of mathematics that treated things perceived by the senses. It accounted for such errors of sight as the apparent convergence of parallel lines, or the appearance of square objects at a distance as rounded.[36] Nonetheless, it was regarded as axiomatic that sensory experience was determined by the senses. That is, although the copies could on occasion be inaccurate, what the senses provided were still veridical copies which were complete and non-divisible.

Secondly, with the assumption of an isomorphism between what was conveyed to the eye and its source in the outside world, the theories were concerned with the problem of how the eye and the mind acquired the qualitative model of visible reality. Whether it was mediated by an *eidolon* or a visual power, the 'copy' was acquired by means of contact. In other words, both theories were characterized by a 'tactile' approach, which explained vision in terms of mechanical contact.

What enabled contact to be established between the eye and the object was the presence of light. Without light, for example, the visual power (ray, *pneuma*) had no means of making contact with the object.[37] Neither type of theory had any conceptual relationship with the physics of light for their treatment of vision. A qualitative sensory 'copy' was not an optical image. As the eye was not conceived of as an organ for 'forming images', the detailed knowledge of its anatomy was basically independent of their explanations of vision just as the detailed anatomy of the hand was independent of any theory of tactile sensitivity. The role of the eye was determined by the teleological assumption that its structure mirrored its function.

Finally, the eye was a perceiving eye. The assumption of a copy excluded the idea that what was conveyed to the eye could be different from what was perceived. As soon as contact was made, perception was direct and complete. The notion of perception as the separate process of interpretation of the sensory record in the sense of a reconstruction of the three-dimensional visual world from a distorted, flat and inverted image in the eye would have been inconceivable. These unifying concepts formed the basis of Islamic approach to vision. They remained essentially unaltered until the introduction of the hypothesis of an optically projected image by Ibn al-Haytham in the eleventh century.

GREEK THEORIES IN ARABIC: CONTINUITY OR TRANSITION?

The Islamic heritage in 'vision' embodied both the theoretical variations of the classical Hellenic positions and the additional arguments in Aristotelian and Pseudo-Aristotelian commentaries of late antiquity based on changing conceptions of space, motion, and time.[38] Along with the visual theories, the Greek mathematical and experimental knowledge in optics and mechanics as well as a detailed anatomy of the eye and its connections with the brain were also acquired through translations into Arabic.[39]

Against this background, it should be mentioned that the aim of Islamic mathematician-astronomers, natural philosophers, and physicians was not only to preserve this legacy, but also to supplement omissions and to correct what were seen as contradictions and errors in, for example, Euclid, Ptolemy, and Galen with increasing emphasis on observational experiments.[40] There were consistent attempts to harmonize Plato and to reconcile Galen with Aristotle on specific issues which are relevant to discussions of vision.[41] It is, in fact, in these criticisms that we may look for the emergence of modifications and further clarifications of visual problems. The question of originality or independent investigation in the Islamic developments is, however, dependent on the nature of the legacy particularly from late antiquity.[42]

Defence of tactile theories: al-Kindī and Ḥunayn ibn Isḥāq

Al-Kindī (d. 866), a major pioneer in the transmission of Greek science, presented a series of arguments against intromission in his optics *(De aspectibus),* which is also a critique of Euclid's theory of vision. Utilizing arguments which were not always new, he clarified some of the major differences between the object-copy and tactile theories.[43] For al-Kindī the validity of a visual theory consisted in being able to account for such problems as the perception of distance, position, clarity, shape and spatial orientation of objects in a way which could be verified by observation and demonstrated by geometrical logic. The intromission or object-copy theory failed to do this.[44]

An inherent strength of the intromission theory was its ability to deal with a common but important feature of everyday perception: namely, our immediate recognition of an object from widely different perspectives as being still the same object, such as a stool as three-legged whether it is viewed from above or from the side. With a holistic copy (as successive 'eidola' or form of the object) entering the eye, the perspective view was irrelevant.

On this question of spatial orientation and the perception of shape, al-Kindī gave the example of a circle viewed edgeways. If vision were the result of a complete form entering the eye, he argued, then the complete circular shape would be perceived. However, viewing the circle edgeways, what is seen was not a circle at all but only a straight line.[45] Therefore, what is perceived was clearly limited by the perspective angle which determined the aspect of the object available to contact with the visual ray. (He did not raise the question that if what is perceived of a circle viewed edgeways is a line, how do we recognize it as a circle at all?) Ironically, in appealing to the arbitration of experience, what al-Kindī had in mind was certainly an ideal rather than an empirical experiment. The extraordinary difficulty we have in seeing the edge of the circle as a straight line could be demonstrated by means of a wire circle (such as one used for making soap bubbles). The slightest head or hand movement that results in lateral deviation will immediately create the perception of the circle. A vast range of oblique views should cause us to see ellipses. In fact, from most positions, we see a circle when it is physically impossible to do so. This constancy in the perception of shape, which was not recognized by al-Kindī, would have been an insuperable problem for the visual ray theory.[46]

Proceeding on the assumption that objects of perception are coherent and indivisible, al-Kindī also raised the objection that if vision occurred by intromission, then irrespective of their position within the field of vision, their distance or proximity, objects would be seen simultaneously and with equal clarity. The eyes need not search to locate them. This was manifestly not the case. In our experience, objects were not perceived all at once but in a temporal sequence, as in reading.[47] He attempted to explain the clarity of visible objects that were near and in the centre as opposed to far away or in the periphery of the field of view on the basis of the weakening of the visual power as it diverged from its source in the eye. Here, al-Kindī did not associate the strength of the central ray of the perspective cone with the shortness of its length as compared to the peripheral rays. Instead, conceiving of the cone as a continuous volume of radiation, his explanation derived from light. Objects located in the centre will be seen more clearly because of the greater concentration of radiation there, just as two candles would illuminate the same area better than one.[48]

Significantly, al-Kindī's arguments for the visual ray were based on geometrical considerations of ideal experiments with luminous sources. On the implicit assumption of the analogy of the visual ray with light, al-Kindī proceeded to demonstrate Euclid's postulate of its rectilinear propagation. In the process, however, he demonstrated the three-dimensional, physical status of rays (as opposed to Euclidean geometrical lines) and their rectilinear propagation from luminous sources.[49] For example, he cites a hypothetical experiment where a candle is placed as a light-source facing an aperture and a screen placed behind it, then a straight line drawn from the outer edge of the illuminated spot on the screen will graze the edge of the aperture and encounter the edge of the candle.[50]

Secondly, in his theory of emission, al-Kindī assumed that from every point on the surface of the eye, rays issue along every straight line that can be drawn from them. His assumption here was also based on a parallel radiation with luminous sources. Thus in al-Kindī, we find not only a series of demonstrations of the rectilinear propagation of light rays, but an explicit description of the radial dispersion of light in all directions from every point on the surface of a luminous body, and illuminating whatever is opposite to which a straight line can be drawn.[51] For al-Kindī, however, as an analogy for the mode of propagation of the visual ray, it constituted the basis of a more precise tactile definition of the visible object within the radiant cone. Its immediate implication is a quantitative or punctate breakdown of the concept of a coherent visual radiation both at its source on the surface of the eye and that of the object with which contact is made. The Euclidean-Ptolemaic cone becomes, therefore, modified into a multitude of cones emanating from every point on the surface of the eye. This provides a three-dimensional trellis of cones which, no matter how far the rays diverged would not allow any object of view to 'slip past' undetected—a major problem for the single cone theories.[52] Although al-Kindī was able to conceive and geometrically analyse a punctate breakdown of the radiation of light, it did not apply to the perceptual world where physical entities were still indivisible.

When al-Kindī turned to the eye itself to further support his position, it was briefly to state that the eyes were not created hollow, like the ears, to collect impressions; they were spherical and mobile so that they could shift their gaze, select the object, and send their rays to it.[53] The underlying assumption was teleological, relating the structure of the eye to its function. Al-Kindī's contemporary Ḥunayn ibn Isḥāq (d. 877), one of the most important translators from Greek and Syriac, used the eye to reject both the intromission theories and the emission of the visual ray.[54] In his *Ten Treatises on the Structures of the Eye, its Diseases, and their Treatment (Kitāb* ***al-ʿashr*** *maqālāt fī* ***al-ʿayn****)* he adopted Galen's theory where, in the presence of light, the *pneuma* transformed the air into an extension of the organ of sight.[55] Ḥunayn describes this transformation in mechanical terms where the *pneuma* emerging from the eye 'strikes' the surrounding air as in 'a collision.' The tactile nature of his understanding of vision is clear in his use of the analogy of the blind man's stick: 'If a person is walking in the dark and holds a stick in his hand and stretches it out full length before him, and the stick encounters an object which prevents it from advancing further, he knows immediately by analogy that the object preventing the stick from advancing is a solid body which resists anything that comes up against it… It is the same with vision.' Consistent with this approach, he attempts to explain how we see things in mirrors and other polished bodies on the basis of deflection, and applies to the Galenic theory, the principle of the equality of the angles of incidence and rebound of the tactile-visual ray theories.[56] With Ḥunayn ibn Isḥāq's *Ten Treatises* and his *Book of the Questions on the Eye* not only a more systematic version of Galen's *pneuma* theory, but also his highly detailed ocular anatomy was transmitted into Arabic.[57]

Neither al-Kindī's principles, nor Ḥunayn ibn Isḥāq's description of ocular anatomy were brought together in the ninth century in spite of their widespread influence. Through Ḥunayn's dissemination of Galen, however, the anatomy of the eye became incorporated into discussions of vision; not only among the physicians and oculists alone who were proponents of the Galenic interpretation, but also among those who rejected the notion of any form of emission from the eye. In fact, anatomy subsequently formed a major part of the attack on the tactile theory in favour of intromission.[58]

Refutation of tactile theories: al-Rāzī and Ibn Sīnā

In *Doubts Concerning Galen (Kitāb fī al-Shukūk ʿalā Jālīnūs),* Abū Bakr Muḥammad ibn Zakariyyā al-Rāzī (d. 923/24) raised the question that if the dilation of the pupil was due to the fact that when one eye is closed, the visual *pneuma* is transferred to the other, how was it that both eyes dilated or became narrow together under different conditions. The change was not due to the internal pressure of the increased *pneuma* at all, as Galen had explained, but to the decrease in external light.[59] He argued that strong light affects sight to the extent of harming it and causing pain whereas in the dark, the eyes cannot see at all. Therefore a compromise was needed between the two extremes, which was provided by the structure of the eye. If the object was in a bright place, the pupil became smaller to allow just enough light for vision to occur and yet prevent damage to sight; if the object was in less light, it became larger to let in a sufficient amount so that vision could occur. Here what al-Rāzī is describing is not the muscular contraction and dilation of the pupil, but the fact that the eye could alter the size of its opening in relation to light. He illustrates the mechanical nature of this process by the analogy of a float or a valve which controls the amount of water in irrigation by increasing or reducing the opening at the mouth of the reservoir so that neither too much nor too little reaches the garden.[60] Thus al-Rāzī considered the pupillary movement as a mechanism which regulates the amount of light that comes into the eye.

In the anatomy section of his *Kitāb al-Manṣūrī (Liber ad Almansorem),* al-Rāzī is more specific: the pupil narrows in bright light and dilates in dim light to provide what was required by the crystalline.[61] The obvious dangers of looking directly at the sun had already been noted in antiquity by Galen and others, but in al-Rāzī we have an unambiguous correlation between the amount of light reaching the eye from the visible object, the change in the size of the pupil, and vision. Unfortunately, al-Rāzī's specific treatise on pupillary movement, and the works on vision which he is reported to have written have not survived.[62] On the basis of the fragmentary nature of the available information so far, it is not possible to evaluate whether he significantly diverged from Hellenistic and Peripatetic ideas on the role of light as a means of conveying (via colour) the coherent form of the visible object and its instantaneous transmission.[63]

The correlation between light, the anatomy of the eye, and vision was taken up by Ibn Sīnā (980–1037) and put to use in the refutation of the tactile theories, both in their geometrical and pneumatic variations. Ibn Sīnā brought together an impressive variety of arguments in numerous works, particularly in his encyclopedic *Kitāb al-Shifāʾ* and its abridgement the *Kitāb al-Najāt,* to demonstrate the logical absurdity of the consideration of emission from the eye to the object, its inconsistency with observable reality, common experience, as well as with the geometry of the visual cone itself in analysing the perception of size and distance of objects.[64]

Reinforcing his ammunition from the arsenals of Hellenistic and Peripatetic refutations, Ibn Sīnā argued that if contact with visible objects is achieved through the base of the visual cone, it necessarily followed that along with their visible qualities, their magnitude will also be communicated irrespective of distance; then the laws of perspective would not apply.[65] The perception of the apparent size was, however, determined by distance with reference to the angle at the apex of the visual cone in the eye. The further away the object, the smaller the angle and smaller the area intercepted by its form on the surface of the crystalline. Therefore, the geometry of the visual cone made sense only if it is considered to originate from the visible object rather than from the eye.[66] Ibn Sīnā gave the example that the same object, held near the eye, forms a smaller angle when moved further away; therefore it will be seen smaller. In fact, sometimes the angle will be so small that the object will fail to be perceived, with the implication that even though it is there still in contact with the base of the cone for the tactile ray to feel/sense it. Thus the angle of the cone is of use as an indication of size in relation to distance and when it is assumed that the 'form' comes from the object to the eye.[67]

Ibn Sīnā's refutations of both the visual ray and the *pneuma* theories impress not so much by their originality, since most of them can be found in writings from Aristotle down to late antiquity; nor by their persuasion in every case, but by the sheer quantity, variety, and comprehensiveness of the arguments deployed to maximum effect.

Ibn Sīnā's own approach to vision is a bare outline in his discussion of sensation as the imprinting of the form of sense objects on the organ of sense. He specifies the conditions for vision: when light falls on the object of sight (a coloured body), which is separated from the eye by a transparent medium (which has no colour), its form is conveyed to the pupil to be imprinted on the surface of the crystalline lens. He goes further to justify intromission on the basis of the anatomy of the eye: 'If this view were not correct, the creation of the eye with all its layers and humours and their respective shape and structure would be useless.'[68] He does not expand on this. What emerges in his description of ocular anatomy in *al-Qānūn fī* **al- Tibb** *(The Canon)* is a definite emphasis, like al-Rāzī, on the incoming light. On the one hand, light needs to reach the crystalline unimpeded, hence the transparency of the aqueous humour and of the extremely fine arachnoid membrane anterior to the crystalline. At the same time, the crystalline has been placed in the centre of the eyeball for protection from [too much] light. Thus the transparency of the layers of the eye, similar to that of the intervening medium, simply allow the light to bring instantaneously, via colour, the visible qualities of [opaque] objects to the crystalline. What is perceived again remains qualitative and indivisible. Ibn Sīnā's repeated reference to what is visible in mirrors as an analogy clearly indicates this traditional conception. He has an elaborate theory of sensation where he distinguishes between the external and the internal senses. The coherent form that sight as an external sense provides is given meaning by the operation of the 'internal senses' which are located in the brain.[69]

Ibn Sīnā's refutations had dispensed with the 'blind man's stick'; reinforcing, instead, the idea of light conveying visual information to the eye, all at once. No explanation was given as to how this phenomenon occurred. It is relevant to note that Ibn Sīnā rejected the mechanical analogy of deflection for light. The grounds for his rejection are revealing: if light were reflected by rebound as in the case of a ball, then it would be reflected from all impenetrable surfaces even if they were not smooth, which by implication is unacceptable for him.[70]

Thus Ibn Sīnā was not in a position to offer a viable theoretical alternative to the concept of coherent forms. Basically, his approach reveals a tactical ingenuity in restating the problems without providing effective solutions, and in appropriating, from the theories which he shows to be inadequate, elements that he can use with added refinements. The result is an eclectic position of encyclopedic scope which, for instance, brings together the Aristotelian concept of 'forms' in sensation, the Galenic ocular anatomy and its connections with the brain as well as the importance of the crystalline in vision, the Peripatetic notion of light as a qualitative 'movement' (i.e. 'leap') from the luminous object to the eye, and the geometrical analysis within a visual cone.

SYNTHESIS OF OPTICS AND ANATOMY: IBN AL-HAYTHAM

Ibn al-Haytham represents a break with the consideration of vision as a qualitative process. In the *Kitāb* ***al-Manāẓir***, he pursued a painstakingly empirical investigation of the properties of light ***(ḍaw')*** as distinct from vision ***(ibṣār)***.[71] At the same time, hegave a highly detailed description of the structure of the eye separately from its function. He then combined both in an attempt to explain vision as the outcome of the formation of an image in the eye due to light.[72]

The hallmark of Ibn al-Haytham's style is an ability to resolve complex issues into independent yet closely interrelated simple investigations, subjecting each problem to a quantitative analysis of its variables under stringently controlled conditions. This is illustrated by his series of experiments on the rectilinear propagation of light, using a dark-room with an aperture in one wall to provide a source of light. The dust in the room allowed the beam to be visualized and tested for linearity. Smoke instead of dust gave the same result. When the room was clear, the light source from the aperture was seen to project a spot of light on the opposite wall. Checked with a rule, followed by further checks, using an interference procedure, it was again found that light travelled rectilinearly. The projected spot of light was only disrupted within a linear path; interference within other paths (such as curvilinear) was ineffective.[73]

Similar experiments were repeated at various times during the day and at night, with different sources of light, using single as well as double compartment dark-rooms with apertures designed to exact measurements. The effect of the size of the aperture was also explored. Furthermore, by means of a tube fixed onto a wooden rule, as a sighting system, with a variable opening, Ibn al-Haytham established that light travelled in straight lines between the visible object and the eye. The successive covering of the opening bit by bit showed a loss of the corresponding parts of the visible object.[74]

He was equally systematic in his investigation of the radial dispersion of light from its source: that light radiates from every point on the surface of an object, whether self-luminous or illuminated, along every straight line that can be imagined to extend from it in all directions.[75] Ibn al-Haytham then determined that this is how light reaches the eye. To investigate whether light radiates from the whole surface of the luminous source, he used not only dark rooms but apparatus constructed to enable exact measurements to be taken. For example, an oil lamp, with a particularly wide wick to provide an intense and constant source of light was placed in front of the opening of a tube where it pierced the centre of a copper sheet to enable the light to be projected via the tube. A screen was placed facing the other end of the tube. When the light was rotated around the aperture (of the tube), the spot projected onto the screen remained unaltered. With the narrowing of the opening, the luminous spot, fainter and smaller, still continued to appear. In this way, he demonstrated that light is emitted equally from all parts of the wick and dispersed radially.[76]

Ibn al-Haytham's exhaustive, comprehensive investigations showed that opaque bodies acquire light from external self-luminous sources (such as the sun). From smooth and polished surfaces light is reflected in a predictable direction (i.e. at equal angles of incidence and reflection); from rough and unpolished surfaces, where some of the light stays 'fixed' *(thabata),* it disperses in all directions from every point along straight lines. What made objects visible was light whether intrinsic or extrinsic to them. Therefore, 'any object to be visually perceived must be either self-luminous or illuminated.' Even transparent bodies, which allowed light to pass through, had some opacity in order to be illuminated and thereby become visible. Thus he established the simple fundamental principle that we see 'ordinary' (i.e. unpolished, non-luminous) objects by the dispersion/radiation *(ashraqa)* of light from their surfaces. This principle underlies Ibn al-Haytham's formulation of a theory of corresponding points between object points and image.[77]

Ibn al-Haytham explained refraction (at both plane and curved surfaces) on the basis of the fact that the speed of light is affected by the density of the medium through which it travels. He separated the motion of light into a vertical component along the normal with a constant velocity, and a horizontal component along the interface of the two media with a variable velocity. In a denser medium (air to water), the velocity is reduced; in a rarer medium (glass to water), it is increased. This principle was utilized by Ibn al-Haytham to explain refraction at the transparent surfaces of the eye.[78]

FROM OBJECT-COPIES TO PROJECTED IMAGES OF LIGHT

Ibn al-Haytham's experiments with images of projected light are crucial for his hypotheses about vision and the eye. Prior to Ibn al-Haytham, images were associated with mirrors or other polished surfaces, including those of the eye.[79] They were explained in terms of either deflection of visual rays or the presence of object copies.[80] By conceptualizing the visual image as an organization of point light sources, Ibn al-Haytham represents a break with this approach to vision as a qualitative process. His notion of light rays projected from each point on the surface of the object to a corresponding point on the screen gives us, for the first time, a simple quantitative explanation of how an image is formed.

We have no direct evidence so far, or knowledge, of image projection via a pinhole in a *camera obscura* prior to Ibn al-Haytham.[81] Although Ibn al-Haytham clearly articulated the principles of a *camera obscura,* pinhole experiments are not described in his *Optics*. His investigations of light involved mostly what would be best described as 'beam-chambers', which were dark-rooms with apertures that allowed beams of light to be projected onto a wall or an opaque surface. The apertures, designed according to specific measurements, could also be progressively reduced in size.[82]

What comes nearest to a *camera obscura* is Ibn al-Haytham's experiment with projection of light via a vertical reduction-slit which is formed by a double-winged door. With several lamps positioned separately on a horizontal plane in front of this opening leading to a dark room *(al-bayt al-muẓlim),* Ibn al-Haytham describes the appearance of patches of light on the wall behind the doors when the opening was reduced to a minimal crack. He noted that if the flame of one lamp were occluded, only the corresponding light patch disappeared on the wall behind the aperture. When the occluder was removed, the patch of light was restored in exactly the same place.[83]

It is worth noting that the experiment is immediately described again in the form of instructions as to how it could easily be repeated. In the second version when the crack in-between the doors is blocked, leaving only a little opening opposite the lamps, Ibn al-Haytham predicts that there will again appear separate [patches of] light on the wall corresponding to the number of lamps, and that each one will depend on the size of the opening.[84]

Ibn al-Haytham's emphasis that the projection depends on the size of the opening is highly significant even though what appears is no more than patches of light, and not a clearly articulated image (i.e. the lamp). Nonetheless, this experiment is not an example of a *camera obscura*. It is still a beam-chamber paradigm with a variable slit as an aperture. In fact, it was used in order to show that separate light rays passed through the aperture in straight lines without interference or mixing, even though they crossed each other, and without affecting the transparent medium (air) through which they travelled. It is significant that Ibn al-Haytham took care to state that the same principle applied to all transparent media, including the transparent coats of the eye.[85]

The originality of this experiment lies in Ibn al-Haytham's use of not just one but a number of lamps providing spatially separated multiple sources of light by means of which he could determine with exactitude the correspondence as well as the inversion of the projection across a horizontal meridian. Repeating from the horizontal to all other meridians would have been both an obvious and a logical progression. There is no doubt that Ibn al-Haytham was able to conceptualize clearly the essential principles of pinhole projection from such an experiment. His subsequent consideration of image-inversion in the eye suggests that at some point such a generalization was made by Ibn al-Haytham.[86]

The multiple light-source projection via a reduction slit provided Ibn al-Haytham with the minimal but sufficient empirical grounds upon which to anchor his theory of corresponding point projection of light from the surface of an object to form an image on a screen. It was an implicit comparison of the eye with a beam-chamber that brought about Ibn al-Haytham's synthesis of anatomy and optics.

THE EYE AS AN OPTICAL INSTRUMENT

Just as Ibn al-Haytham systematically investigated the propagation of light independently of its effect on the eye, he gave a highly detailed description of ocular anatomy independently of his hypothesis about image formation in vision. It was only after clarifying the structural organization of the eye that he showed its functional significance as an optical system. Thus Ibn al-Haytham for the first time deals separately with what could appropriately be termed as 'descriptive' and 'functional' anatomy of the eye.[87] As Ibn al-Haytham's 'eye' has frequently been misrepresented, it is necessary to provide a detailed description close to the Arabic text.[88]

Descriptive anatomy

Taking the eye as a direct outgrowth of the brain, Ibn al-Haytham begins with the optic nerves as consisting of two distinct sheaths which originate from the membranes of the brain. They emerge from the sides of the anterior part of the brain and come together to form the optic chiasm (the common or the joint nerve in the midline). Diverging again, they go separately to the socket of each eye. The 'hollow' optic nerve enters this socket via the optic foramen and expands to form the eye itself. The globe of the eye is contained within the cavity of the bony orbit. The space between the cavity of the orbit and the globe of the eye itself is filled with a substantial layer of fat.[89]

In a tightly organized logical progression, Ibn al-Haytham considers each component part of the eye. First of all, the extension of the external sheath of the optic nerve forms the sclera as well as the cornea. Secondly, the inner sheath constitutes the 'uvea', or the grape-like layer, which includes the iris, the ciliary body and the choroid coat. Although this is faithful to the gross anatomy of Galen, there are already significant differences in approach.[90] For instance, the whole area of the eye behind the iris (corresponding to the combined posterior and the vitreal chambers of the eye), constitutes what Ibn al-Haytham uniquely calls the 'uveal sphere'.[91] The anterior of this dark sphere is perforated by a round hole which is directly opposite the funnel of the optic nerve. The pupil and the uvea are covered by the cornea, a tough, transparent layer which is a modified continuation of the sclera.[92] Both the outer and inner surfaces of the cornea are carefully described as being parallel because of an even thickness throughout. The space both in front and behind the iris is filled by the transparent aqueous fluid, which has the consistency of albumin. This fluid is in contact with the inner concave surface of the cornea, and through the pupil with the anterior surface of the lens, a description which allows for the anterior and the posterior chambers of the eye.[93]

The lens is immediately behind the pupil. It is described as a small body and called 'ice-like' because of the nature of its transparency.[94] Its anterior surface, similar to the outside of a lentil, is flattened following the curvature of the uvea.[95] Behind the lens is the vitreous humour, or the glass-like fluid. The nerve, which expands like a funnel enclosing the vitreous, is attached to both the ciliary body and the lens at its equatorial perimeter. Ibn al-Haytham regards the lens and the vitreous also as a unit consisting of two parts with different transparencies. The basis of this consideration is their combined shape which is spherical.[96]

Furthermore, the fluid parts, such as the aqueous, the lens, and the vitreous are contained by the various layers which define and maintain their spherical shapes. For example, the aqueous is contained not only by the cornea and the uvea (the ciliary body and the iris) but [caudally] also by an extremely fine membrane, the 'arachnoid'. This membrane in turn surrounds the lens and the vitreous. The globe of the eye within the socket is maintained by the sclera.[97]

At the same time, some of the typical elements which characterize Galenic ocular anatomy are certainly here: such as the 'hollow' optic nerve, the optic foramen sited opposite the pupil instead of being nasally displaced, the lens in direct contact with the vitreous, the existence of an 'arachnoid' membrane.[98] In addition to specific differences, Ibn al-Haytham provides an unvarnished account, devoid of the kind of teleological exposition of structure or the qualitative humoural theory with its attendant temperaments which were an inseparable part of traditional anatomy.[99] It is characterized by a single-minded concentration on the shape, position, and condition of the parts of the eye, with great emphasis that these remain constant, and that their inter-relationships are stable.[100]

Secondly, after establishing how the eye is constructed, Ibn al-Haytham makes his most original contribution by a detailed consideration of the functional significance of this anatomy as an optical system. This is exemplified by his description of the lens and the axis of the eye.

Functional anatomy

Unlike the previous references to the lens simply as anteriorly 'flattened' or 'lenticular', Ibn al-Haytham provides an exact description of its 'biconvex' shape on the basis of the difference in the radial lengths of its anterior and posterior surfaces.[101] It is clearly stated that the anterior surface of the lens is part of a spherical surface greater than the spherical surface of its remaining portion (i.e. its posterior surface).[102] The two surfaces of the lens are considered as belonging to different spheres, one larger than the other (see Figure 20.1). This means that the anterior curvature of the lens, if extended, would circumscribe the back of the eye and constitute the circumference of the larger sphere, enclosing the lens and vitreous. The larger sphere would then comprise the lens and the vitreous. Such an analysis is entirely consistent with Ibn al-Haytham's earlier description of the spherical shape of the lens and the vitreous when taken together as a unit or single entity. It is also congruent with his unique conception of the 'uveal sphere' as the whole area of the eye behind the iris, again containing the lens and the vitreous humour.[103]

The shorter radial curvature of the posterior surface of the lens, on the other hand, would be continuous with the anterior surface of the cornea.

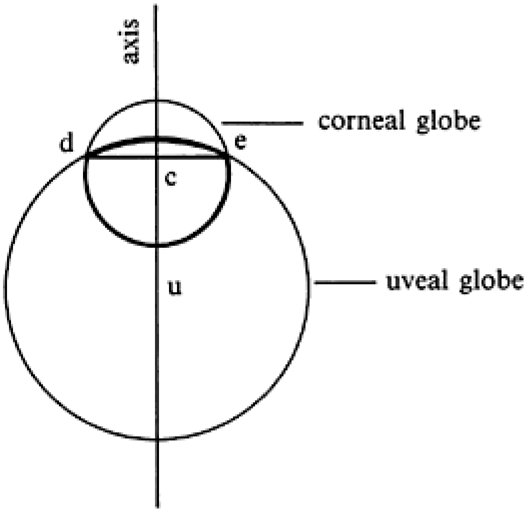

Figure 20.1 Ibn al-Haytham's 'eye', consisting of the intersection of two spheres of differing sizes, one small and one large, where the area of their intersection is the lens. The bisecting perpendicular to the chord of the intersection is the optical axis; (c) indicates the corneal centre and (u) the uveal centre.

The smaller sphere would then be composed of the lens and the cornea. This is also supported by his description of the concave inner surface of the cornea intersecting the convex uvea and establishing a continuity with the posterior surface of the lens.[104] The two spheres constituted in this way intersect at the junction of the ciliary body and the lens. Their relative position is also indicated by the difference in their radii, where the centre of the larger sphere is described as deeper within the globe of the eye than that of the smaller one.[105] This analysis is entirely consistent with his descriptive anatomy (Figure 20.2).

In this way, Ibn al-Haytham describes an eccentric intersection of the segments of two spheres of differing sizes, one small and one large, where the area of their intersection is the lens. It is not a concentric ' onion-layered eye'. The two surfaces of the lens are in fact described as intersecting spherical surfaces.[106] With this analysis the location of the lens is unequivocally identified in its forward position towards the cornea (see Figure 20.2). The centre of the eye naturally falls behind the lens in the vitreous humour, and becomes the centre of the larger 'uveal' sphere.

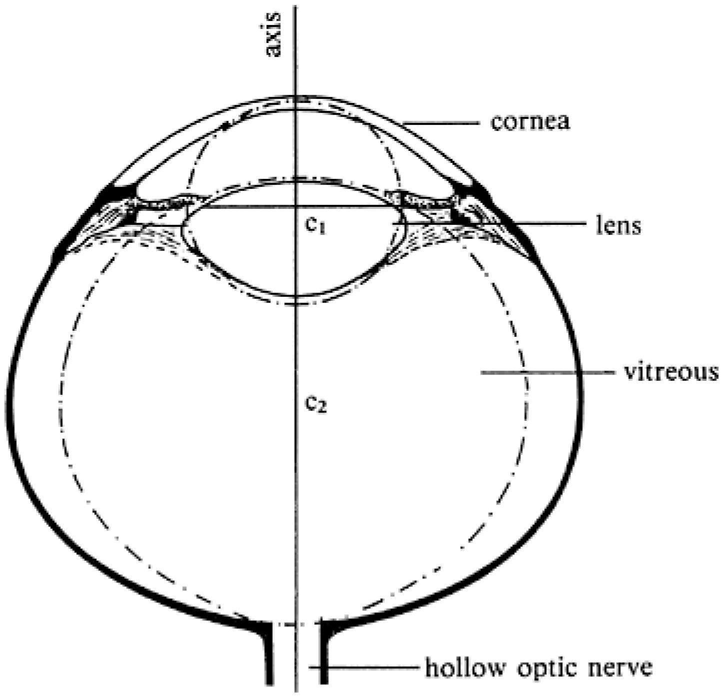

Figure 20.2 A schematic of the eye in sagittal section. The dashed line drawing of Ibn al-Haytham's double-globed eye is superimposed to illustrate the accuracy of his anatomical description. The optic nerve is, however, directly opposite the pupil instead of its correct nasal displacement.

This structure also enabled Ibn al-Haytham to formulate an axis for the eye by connecting the separate centres of the two spheres by means of a straight line which is perpendicular to the chord of intersection of the two spheres, bisecting it at right angles (illustrated in Figure 20.1). The defining characteristics of this axis are carefully enumerated by Ibn al-Haytham: it passes through the centre of the globe of the eye, and if extended at its two extremes, goes through both the centre of the pupil and the centre of the funnel of the optic nerve.[107] His functional definition is again determined by his descriptive anatomy where the optic nerve was placed directly opposite the pupil, instead of its slightly nasal displacement. It erroneously aligned the centre of the posterior curvature with the centre of the optic nerve. Ibn al-Haytham's pioneering attempt to determine an axis for the eye, though expressed in geometrical terms, is here determined by inherited anatomical assumptions from the Galenic tradition.

The definition of this axis is crucial for Ibn al-Haytham's quantitative approach to image formation on the basis of corresponding points. It serves as an optical axis upon which all the refracting media (the cornea, the aqueous, the lens, the vitreous) are centred. By means of this axis the correspondence of the topological location of each point between object and image can be maintained through convergent eye movements (where the axes of the two eyes converge on a point on the surface of the object) and conjugate eye movements (where the axes of the two eyes move together) in changes of gaze from one object to another.[108]

In further defining this axis, Ibn al-Haytham frequently changes his terms of reference from spheres to surfaces, considering the eye longitudinally as well as coronally (see Figure 20.3). This distinction is extremely important because in each case the set of relationships which are depicted lie on different anatomical planes. When he considers the eye longitudinally, the centres of the parts of the eye are aligned along the sagittal axis (Figure 20.3a). When he compares the relative positions of the cornea, the iris, the pupil, and the lens with regard to this axis and insists that their centre is one and the same, he is viewing the eye coronally where the centres (though placed one behind the other along the axis) appear to be at a single point (as in Figure 20.3b). For example, although the radius of the cornea is longer than that of the iris, their centre remains the same. That is, they have different radii which appear to come from the same centre which is on the longitudinal axis of the eye (as in Figure 20.3b).[109] The failure to distinguish the changing perspective of these separate anatomical planes has been responsible for the misinterpretation of Ibn al-Haytham's emphasis on a common centre. It is the mixing of both the sagittal and the coronal planes on the same level (i.e., the axis passing through a single point and the

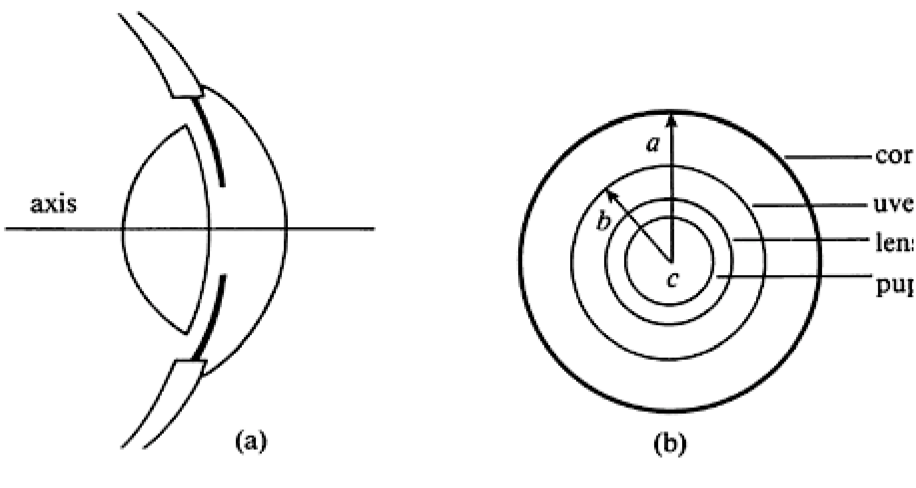

Figure 20.3 Schematics of the eye in different anatomical planes. The sagittal view (a) with the centres on the axis, and the coronal view (b) of all the centres at a single point; *a* and *b* indicate the radial lines.

centres at a single point) which has given rise to the misconception of the mediaeval concentric 'onion eye' that is attributed to Ibn al-Haytham.[110]

His treatment of ocular anatomy is distinguished by a straightforward factual description of the parts of the eye in a tightly organized logical progression together with a detailed analysis for the first time, as far as we know, of their spatial relationships in functional terms for physiological optics. The originality of his anatomical method initiates a decisive departure from the traditional approach. It is neither idealized beyond a description in geometrical terms nor adjusted, as it has been assumed, to accommodate the necessities of a theoretical position.[111] His functional analysis is entirely based on his descriptive anatomy, which was more accurate than those in medical texts. By close attention to structural proportions, he was able to clearly describe the biconvexity of the lens and identify its forward position. By formalizing his description quantitatively, i.e. in proportional terms, he was able to define an optical axis for the eye. This powerfully demonstrates how the central axiom for his physiological optics was firmly anchored in anatomical considerations.

THE PROJECTED IMAGE AND THE EYE

Ibn al-Haytham's hypotheses about vision and the eye could be interpreted as a series of attempts to reconcile his notions of image projection with the anatomic structure of the eye. Such a model confronted him with profound conceptual and technical difficulties when applied to the eye with its large aperture, the pupil, and its transparent refractive surfaces. Furthermore, there were two images to contend with, one in each eye, when our perception of the world was unitary.

Problem one: Size of aperture—the pupil

On the basis of his experience with variable apertures, Ibn al-Haytham was fully aware that projection from a light-source into a dark chamber depended on the size of the opening and that it was only with a minimal opening that a distinct image (of light) could be obtained.[112] Reducing the aperture to its minimal size acted as an exclusion device, filtering out the multiple light rays from each point on the surface of an object and allowing only one ray to pass through to result in a point to point correspondence (see Figure 20.4a). Otherwise when each object point had a multiple representation (with a widened aperture), the pattern of rays would have been degraded to an indistinct patch and lost as an image (see Figure 20.4b).

This was precisely the problem with the eye: its aperture, the pupil, was too large to filter out the multiple rays simultaneously reaching it from each point on the surface of any visible object. The question then was how to maintain a point to point correspondence between the object and the eye.[113] Although Ibn al-Haytham described the crystalline humour as a biconvex refractive body, he did not treat it as a lens capable of providing a point-focus function in the eye. Therefore the solution Ibn al-Haytham found derived initially from mechanics rather than refractive optics. On the basis of experimental observations, he had concluded that it was only the impact of perpendicular projectiles on surfaces which was forceful enough to enable them to penetrate whereas the oblique ones were deflected. For example, to explain refraction from a rare to a dense medium, he used the mechanical analogy of an iron ball thrown at a thin slate covering a wide hole in a metal sheet. A perpendicular throw would break the slate and pass through, whereas an oblique one with equal force and from an equal distance would not. He also knew from his observations that intense direct light hurt the eye. Applying mechanical analogies to the effect of light rays on the eye, Ibn al-Haytham associated 'strong' lights with perpendicular rays and 'weak' lights with oblique ones. The obvious answer to the problem of multiple rays and the eye was in the choice of the perpendicular ray since there could only be one such ray from each point on the surface of the object which could penetrate the eye.[114]

Perpendicular rays: The filter principle

By emphasizing only those which were perpendicular to the surface of the eye, Ibn al-Haytham thereby excluded all incidental or oblique rays. Thus from each point on the

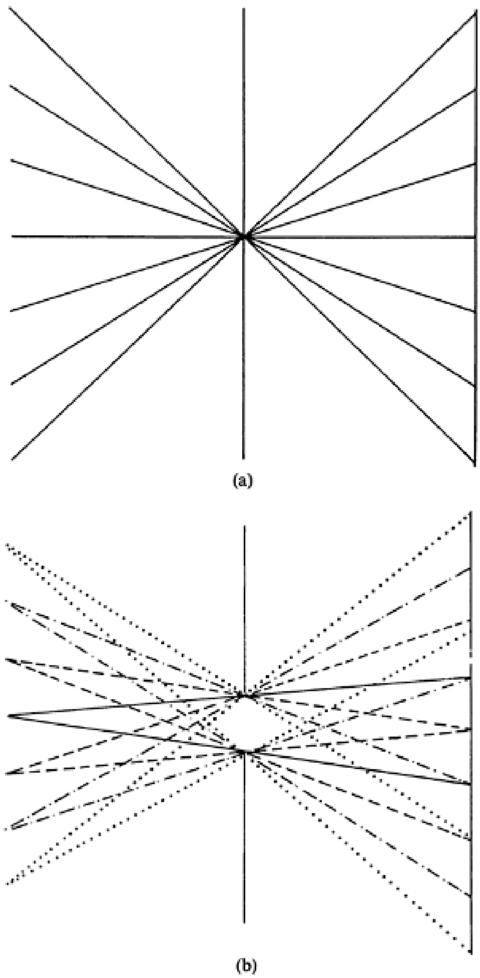

Figure 20.4 Light projection via a pinhole (a) and an aperture (b). In figure (a), each object point is represented by a single ray; whereas in figure (b), each point has multiple representation.

object, only one direct ray would penetrate the layers of the eye and one set of these 'single' rays would preserve the order of their points of origin from the surface of the object. In this way, there would be a point to point correspondence between the visible object and the image in the eye. What Ibn al-Haytham proposed was in effect an alternative method of filtering out the multiple rays from each object point to a single one (as opposed to either a pinhole or a point-focus by the lens).

Ibn al-Haytham had provided the key elements for this hypothesis in his functional analysis of the anatomy of the eye. His intersecting two-globe eye, with the lens as the area of their intersection, demarcated the cornea as the segment of the small globe, and the anterior surface of the crystalline lens as that of the large globe. A projected sagittal line through the two spherical centres of the small corneal and the large uveal globe enabled him then to give an exact definition of an axis, on which all the refractive transparent surfaces were centred and which was perpendicular to all the surfaces of the eye. By means of this axis the correspondence between the topological location of each point on the surface of the object and that of the eye could now be determined and maintained.

In his closely argued position, the essential steps are clearly set out. First of all, 'the nature of sight is to receive what comes to it of the form (i.e. light and colour) of visible objects' and to receive 'only those forms that come to it along certain lines.' Secondly, 'it has also been shown that the form of each point on the visible object reaches the eye opposite in many different lines, and that the eye cannot perceive the form of the object with the order it has on the surface of the object, unless the eye receives the forms through the straight lines that are perpendicular to the surface of the eye and to the sentient organ (i.e. the lens).' And thirdly, 'it has been shown that the straight lines cannot be perpendicular to these two surfaces unless their centres are one common point…' Here, the reference is to the eye viewed coronally, where the two spherical centres, the centre of the corneal surface and that of the lens converged at a single point (i.e. on the axis); in other words, with their different radii coming from the same centre (as in Figure 20.3b). Therefore, '…the eye does not perceive any of the forms coming to it except through the straight lines which are imagined to exist between the visible object and the centre of the eye and which are perpendicular to all surfaces and layers of the eye' (see Figure 20.3a).[11]

The basis of his choice of perpendicular rays is also clearly stated: 'But the effect of the lights that come along the perpendiculars is stronger than the effect of those that come along oblique lines. Therefore, it is appropriate that the lens should sense, through each point on its surface, the form that comes to this point along the perpendiculars alone, without sensing through the same point that which comes to it along refracted lines'.[116] Thus his concern is with a point to point correspondence. And in excluding the weaker incidental rays, the underlying principle is a simple intensity filter which derived from the notion of mechanical impact.

The sensitivity of the lens

Ibn al-Haytham's observations of the effect of strong light on the eye not only reinforced his intensity-filter principle, but also enabled him to explain visual sensation as an experience analogous to that of pain. Intense light gave rise to pain whereas less intense forms of light generated less awareness.[117]

For Ibn al-Haytham, the lens, being 'ice-like' or crystalline, was a transparent body which allowed the light to go through according to optical principles, but at the same time being somewhat dense to detain it long enough for sensation to register. Therefore, it was distinguished from other transparent media that only transmitted light without being affected by it.[118] Since Ibn al-Haytham associated the effect of light on the lens with a range of awareness from indiscernible to sharp pain according to the amount of light, the sensitivity of the lens was in providing information on the impact/intensity of light. This is borne out also by the fact that he pays a great deal of attention to the importance of the function of the iris and the uvea in providing a dark and opaque surface within the uveal globe of the eye, a dark chamber, where even the weakest/faintest light and colour can show up.[119]

Problem two: The inversion of a projected image

The lateral image inversion that Ibn al-Haytham demonstrated with the lamp-experiment provided him with a conceptual model of corresponding point-projection of visual images. It was also an empirical demonstration to him that such projections would inevitably be inverted due to intersection of light rays passing through a small aperture. This meant that in applying such a model to vision, the inversion of the image (both horizontal and vertical) had to be reconciled with the veridical perception of the normal (i.e. upright) world.

Attempted solutions: The optical mechanics of the eye

Ibn al-Haytham's view of the eye as segments of two intersecting globes is essential to his explanation of image projection in the eye. In defining the rays which were responsible for corresponding points, Ibn al-Haytham insisted that they must be perpendicular to the surfaces of both the cornea and the crystalline lens, and he also identified their path with the radial lines from the coronal centre of the eye. It is within the anatomical structure of the eye that the significance of this insistence emerges. Furthermore, the reasons underlying his definition of these rays with reference to radial lines become comprehensible when the separate centre of each intersecting globe is considered. The embedded logic of Ibn al-Haytham's position can be clarified if we reconstruct the steps in his theoretical formulation of image projection in the eye (see Figure 20.5).

Viewing the eye in the sagittal plane, it can be seen that (1) a ray which is perpendicular to the cornea (i.e. radial to the centre of the smaller/corneal globe) will be incidental to the anterior surface of the lens (Figure 20.5a). As an incidental ray, it would be too weak to pass through the lens to form an image. (2) Conversely a ray that is perpendicular to the surface of the lens (i.e. radial to the centre of the larger/uveal globe) will be incidental to the cornea (see Figure 20.5b) which again means that it would be too weak to be penetrate. (3) In order to form an image, a ray must therefore be perpendicular to both the cornea and the lens. The only way this can occur was by means of refraction (see Figure 20.5c). The ray which is perpendicular to the corneal surface (i.e. radial to the first centre) is refracted at the anterior surface of the lens to the perpendicular to pass through the second radial centre of the 'uveal' globe. (4) Although the rays are now perpendicular to both surfaces, when they pass through the centre of the 'uveaP globe of the eye, they would diverge, and the image would be inverted at the back of the eye. (5) Since an inverted image is at variance with the perception of an upright world, it could be neither true nor veridical. Therefore, Ibn al-Haytham proposed a second refraction at the posterior surface of the lens. Due to the difference in the optical density of the lens and vitreous, this refraction would be away from the axis, towards the normal to prevent the intersection of rays at the centre, preserving the upright orientation of the image at the back of the eye (see Figure 20.5d). (6) Thus having preserved both its order of corresponding points and its veridical orientation, the pattern of light rays is projected to the cavity of the hollow optic nerve to be conducted to the chiasma or the common nerve.

By this means Ibn al-Haytham provides an elegant solution to problems of image formation in the eye in bringing together optics and anatomy. Although his answers were erroneous, nonetheless for the first time, an explanation was given of the dioptric mechanism underlying the function of the parts of the eye.

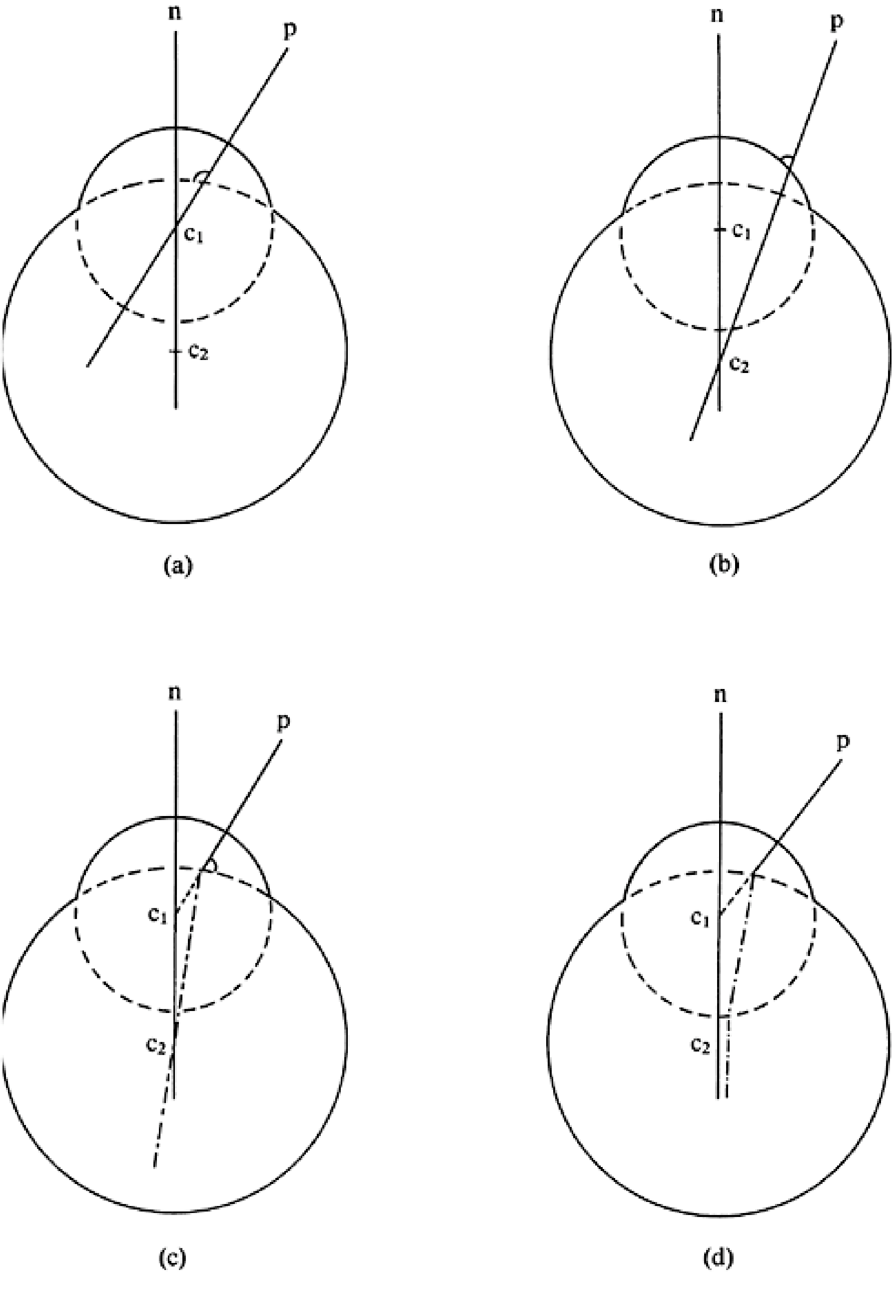

Figure 20.5 Schematic representation of Ibn al-Haytham's views on image formation in the eye, using the principles of perpendicular rays and refraction; (n) indicates the normal, and (p) the perpendicular. (See text for details.)

Refraction: Filter principle extended

It is important to recognize that Ibn al-Haytham did not put forward an inflexible theoretical position concerning image formation in the eye. Rather, he continuously developed his hypotheses as his empirical optical acquaintance expanded. When he experimentally discovered that incidental rays also carried visual information to the eye, he modified his position. For example, he noted that a small object, such as a needle or a pen, held close to the temporal corner of one eye while the other eye is shut, can be seen. Since no perpendicular line could be drawn from the object-point in this position to the surface of the eye, it must be seen by refraction. Again, a small object such as a needle, placed close to one eye, while the other is shut, did not hide an object-point lying directly behind it on the common line (axis) drawn from the centre of the eye. Since the object point could only be seen by an oblique ray, it must be refracted at the surface of the eye. He further noted that the needle appeared to be wider and transparent, enabling us to see behind it. Fine marks put on the wall were fully visible and not occluded by the needle, when close up. From such observations, Ibn al-Haytham was forced to conclude that the only way visible objects could be perceived was by means of refraction, with the full awareness that it had neither been noticed nor explained before him. [120]

Ibn al-Haytham's emphasis that 'we see by refraction' to the extent of regarding it as his 'original contribution' (in Book VII of his *Optics*) may seem at variance with his total exclusion of refracted rays (in Book I). It is in fact an important extension of his principle of filtering on the basis of perpendicular rays. In incorporating refraction into his hypothesis of image formation, Ibn al-Haytham still maintained his intensity-filter principle. He had established that the only way the optical system of the eye could filter the multitude of rays from each object point was on the basis of the perpendiculars. Therefore to be able to preserve point to point correspondence between object and image, he again considered as effective only those which were refracted to the perpendicular. In this way, he excluded all other incidental rays. Refraction to the perpendicular at the anterior surface of the cornea and the crystalline lens was defined with reference to their spherical centres. Hence, they were perceived [as though] along the radial lines from the coronal centre of the eye.

What he proposed is not contradictory at all. It is a shift from his initial absolute intensity position where only direct perpendicular rays were effective to that of relative intensity, incorporating certain incidental rays; that is, only those that were refracted to the perpendicular. The common denominator of his hypothesis of corresponding points is still the perpendicular rays. His incorporation of refraction does, however, represent a major step from a mechanical solution of the projected image in the eye towards an optical one.

Problem three: Diplopia

(Duplexity of the images and the unity of visual experience)

Central to any attempt to understand the physiological mechanism of vision is the requirement to account for the subjective experience of perceptual unity. In other words how is it that we have a single percept when the possession of two eyes should result in double vision or diplopia.

Ibn al-Haytham's solution to this problem relied on a precise quantitative matching of the sensory information from each eye. Each pattern, preserving its spatially organized information, travelled via the channel of the optic nerve to be integrated in the common nerve before reaching the anterior part of the brain.[121] Although it is not clear to what extent this process is according to optical principles, Ibn al-Haytham's discussion of the imperceptible speed with which the pattern of sensations reached the chiasma suggests an analogy with light transmission in a beam chamber. '…[It] arrives at the chamber of the common nerve in the same way light arrives from windows and apertures, through which light comes, at the surfaces (i.e. walls or screens) opposite these openings'.[122] Here too Ibn al-Haytham depicted a point by point projection and superimposition of the two images coming from the two eyes. In other words, the separate punctate templates were combined in the 'common nerve' and, if they made an exact match, fused into a single record.[123]

This notion of image 'fusion' or matching in the common nerve is empirically supported by Ibn al-Haytham's awareness of the critical role of eye movements for binocular integration. Equal horizontal convergent movements were essential in order to maintain the correspondence of the image in each eye. Conjugate eye movements during changes of gaze from one object to another or even in different regions of the same object served the same function. For example, when the observer looked at the visible object, by directing the pupil towards it, the axes of the two eyes converged at some point on its surface; when the observer moved his eyes over the visible object, the two axes moved together over every part of its surface. It was impossible for one eye to be moved towards a visible object and for the other to remain at rest unless restrained.[124]

When the two images were spatially misaligned, i.e. by having the observer fixate an object with one eye while deviating the other eye, then double vision or diplopia was produced. Being topologically out of register, due to the mispositioning of the two images in the eye, no fusion would occur in the chiasma, resulting in double vision. Ibn al-Haytham does not seem to have considered the disparity of the 'identical' images in each eye as a basis for stereopsis, or depth perception.[125]

CONCLUSION

Ibn al-Haytham showed that the object itself is not sensed at all, but that innumerable points of light deflected from the surface of the object to the eye resulted in the sensing of an 'image' which is formed according to optical principles. Within the Greek legacy, Ibn al-Haytham's approach to vision was a conceptual change which destroyed the viability of that tradition.

He has left us with the fundamental transition from the tactile mechanics of vision to a corresponding point theory of image formation due to light alone. Although his experimental investigation of both reflection and refraction still derived from the principles of mechanics, his work is the springboard of all subsequent optical studies of vision.

Due to Ibn al-Haytham, anatomy of the eye emerged, from being a passive or active guest in discussions of vision, as an essential equal partner with optics because an understanding of vision increasingly required a synthesis of anatomy (biology) with the physics of light. Thus physiological optics owes its existence to this union. Subsequently the visual concern shifted from the global question of 'how do we perceive the external world by the sense of sight' to specific problems arising from the implications of a punctate optical image in the eye: (a) the preservation of a point to point correspondence between object and image; (b) image inversion and the veridical (upright) perception of the object; (c) the unity of perception, or the binocular fusion of the two separate images, one from each eye; (d) the distinction between the image as a two-dimensional pattern in the eye and its perception as a three-dimensional object by the mind/brain.[126] These became central issues, defining physiological optics, to Descartes and beyond.

There is no evidence so far that the implications of Ibn al-Haytham's corresponding point theory of image formation were taken up in Islamic science with the exception of Kamāl al-Dīn al-Fārisī (d. c. 1320).[127] In his *Tanqīḥ al-Manāẓir*, based on Ibn al-Haytham's work, Kamāl al-Dīn went further in experimental investigations of the role of incidental rays in image formation in the eye. For example, he correctly demonstrated that the 'pupillary picture' which had been associated with the lens was in fact an image reflected mainly from the cornea with a second, fainter one from the lens. He also observed, through the pupil, the image on the lens of a freshly slaughtered sheep's eye.[128] The contribution to physiological optics of his studies in image formation, stereopsis or depth perception, and other areas still awaits research.

The extent to which Ibn al-Haytham is individually responsible for the divergence from the classical world cannot be determined in this paper. Nor are we in any position yet to establish the sources of his originality. It would be rash to discount the possibility powerful and lost influences on his work. There are intimations of divergent thinking in the period just before the Islamic era. Future research may provide important clues to Ibn al-Haytham's creativity also by bringing to light other works by his immediate predecessors as well as his contemporaries. The paradigm shift in the *Kitāb al-Manāẓir*, is attributed to Ibn al-Haytham with the qualification that there are gaps in the existing record. What is beyond doubt is that the *Kitāb al- Manāẓir* represents the earliest extant formulation of this crucial change in thinking about vision.

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NOTES

1 S.Polyak, *The Vertebrate Visual System,* iii (Chicago, 1957), esp. 147–52.
2 Polyak, *ibid.,* ii, esp. 100–104; Descartes, *Traité de l'Homme* (1664), reprint, and trans. by S.T.Hall, *Treatise of Man* (Cambridge, 1972), 83–86.
3 *Kepler: De Modo Visionis,* trans. A.C.Crombie in *Mélanges Alexandre Koyré: L'aventure de la science,* I (Paris, 1965), 135–172; F.Platter, *De Corporis humani structura et usu…libri III* (Basilea, 1583), p. 187; D.C.Lindberg, ' Johannes Kepler and the Theory of the Retinal Image,' in *Theories of Vision From al-Kindi to Kepler* (Chicago, 1976), 193–205.
4 The psychology of perception falls outside the main concern of this paper and deserves a separate study. See, G.Hatfield and W.Epstein, The Sensory Core and the Medieval Foundations of Early Modern Perceptual Theory,' *Isis,* 70 (1979), 363–84.
5 For Aristotle, the sense of touch gets its name from the fact that it operates through direct contact, *De Anima* (435a17–18); for a discussion of the contact-criterion, see R.Sorabji, ' Aristotle on Demarcating the Five Senses,' in *Articles on Aristotle,* IV, v (Psychology and Aesthetics), ed. J.Barnes, M.Schofield, R.Sorabji (Duckworth, 1975–9), particularly 85–92.
6 For a discussion of the visual theories in antiquity and relevant bibliography see A.C.Crombie, *The Mechanistic Hypothesis and the Scientific Study of Vision: Some Optical Ideas as a Background to the Invention of the Microscope.* Repr. from Proc. of the Royal Microscopical Soc., II, i (Cambridge, 1967), 3–16; also 'Early Concepts of the Senses and the Mind,' *Scientific American,* vol. 210, no. 5 (May, 1964), 108–16; and Lindberg, *Theories of Vision,* 1–18; also Lindberg (1978a).
7 For a discussion of *eidola,* see Edward N.Lee, 'The Sense of an Object: Epicurus on Seeing and Hearing' in *Studies in Perception: Interrelations in the History of Philosophy of Science,* ed. Peter K.Machamer and Robert C. Turnbull (Columbus, Ohio, 1978), ii, 27–59, esp. 27–29.
8 Euclid's Definitions (1–7) and propositions I–VIII clearly set out a geometrical analysis of vision with reference to a visual cone, see M.R.Cohen and I.E. Drabkin, *A Source Book in Greek Science* (Cambridge, Mass., 1948), 257–58.
9 Although the analogy of the ' stick' was explicitly made by the Stoics, the idea of textile extension is clearly expressed by one of Euclid's followers, the mathematician-astronomer Hipparchus, who compares the visual rays to hands reaching out to the object, see D.E.Hahm, 'Early Hellenistic Theories of Vision and the Perception of Color' in *Studies in Perception,* iii, 79.
10 On visual ray theories of Euclid and Ptolemy, see Albert Lejeune, *Euclide et Ptolémée: Deux stades de l'optique géométrique grecque* (Louvain, 1948), and his critical ed. *L'Optique de Claude Ptolémée, dans la version latine d'après l'arabe de l'émir Eugène de Sicile* (Louvain, 1956). For their summary with relevant bibliography, see Lindberg, The Mathematicians: Euclid, Hero, and Ptolemy,' in *Theories of Vision,* 11–18.
11 For the criticism of intromission theories on the question of 'size' with reference to the 'mountain', see Galen, *De Placitis Hippocratis et Platonis.* (On the Doctrines of Hippocrates and Plato), ed. and trans. P.De Lacy (Corpus Graecorum Medicorum) (Berlin, 1978), VII, 5. 2–5; 7–8.

12 For the theory of reflection in mirrors, see Hero, *Catoptrics,* 1–7, 10, 15, and Ptolemy, *Optics,* III, *A Source Book in Greek Science,* 261–71.

13 For a comparison of reflection in vision and mechanical deflection, see the Peripatetic *Problemata,* XVI, 13, 915b in C.B.Boyer, 'Aristotelian References to the Law of Reflection', *Isis,* 36 (1945–46), 94.

14 For Galen's criticism, see *De placitis Hippocratis et Platonis,* VII, 5. 2–6, trans. De Lacy.

15 For Plato's discussion of vision in the *Timaeus,* 45b–d (trans. F.M.Cornford, *Plato's Cosmology,* 152–57) and *Theaetetus,* 156 d-e (trans. F.M.Cornford, *Plato's Theory of Knowledge,* 47), see Crombie, *The Mechanistic Hypothesis,* nt. 9, 6–7; and Lindberg, *Theories of Vision,* 5–6.

16 The tactile basis of Plato's emission theory is also briefly discussed by Hahm, 'Early Hellenistic Theories,' 71–75.

17 Diogenes Laertius, vii, 157 quoted in Crombie, *Mechanistic Hypothesis,* p. 8, nt. 11; for the accounts of the Stoics, see S.Sambursky, *The Physics of the Stoics,* (London, 1959), 21–29, 124; and particularly Hahm, 'Early Hellenistic Theories', 65–69.

18 Hahm, 'Early Hellenistic Theories', 85.

19 *De usu partium,* trans. M.T.May, *Galen, On the Usefulness of the Parts of the Body,* II (Ithaca, 1968), x, 1, 463–64; *De Placitis Hippocratis et Platonis,* VII, 6, 28–29, trans. De Lacy.

20 *De placitis Hippocratis et Platonis,* VII 3.10–6, 4.17 trans. De Lacy; and *De usu partium,* II, x, 463–503, trans. May. For a full account of Galen's theory in relation to his predecessors and the importance of his anatomy, see R.Siegel, *Galen on Sense Perception* (Basel, 1970); Galen's theory as a synthesis Plato, Aristotle, and the Stoics, see H.Chernis, 'Galen and Posidonius' Theory of Vision,' *American Journal of Philology,* 54 (1933), 154–61.

21 For a discussion of the 'walking stick' analogy in relation to the nerve in Galen, see *De placitis Hippocratis et Platonis,* VII, 5.5–11, 5.40–41; 7.16–8.22, trans. De Lacy.

22 For a discussion of these two views in Galen, see Robert J.Richards' review of Lindberg's *Theories of Vision* in *Journal of the History of Behavioural Sciences,* XV (1979), 378–82.

23 For a definition of sensation in Aristotle, see Charles H.Kahn, 'Sensation and Consciousness in Aristotle's Psychology' in *Articles on Aristotle,* IV, esp. 3–5.

24 *De Anima.* II, 6, 12, trans. R.D.Hicks in *A Source Book in Greek Science,* 542–43.

25 For a clarification of Aristotle's views on vision, particularly his definition of sight with reference to its 'objects of sense', see R.Sorabji, 'Aristotle in Demarcating the Five Senses,' *Articles on Aristotle,* 76–92, esp. 77–85.

26 Galen was clearly aware that Aristotle did not develop a theory of vision which could explain 'how we distinguish the position or size or distance of each perceived object,' *De placitis Hippocratis et Platonis,* VII, 7.4–15, trans. De Lacy. He is, in fact, rather harsh in his criticism of Aristotle for utilizing emitted rays when discussing 'the things seen through mirrors.' (VII, 7.10–16) For Aristotle's assumption of the visual ray in his *Meteorologica* in constrast to his views in *De anima* and *De sensu,* see Boyer, 94–95; and Lindberg, *Theories of Vision,* 217, nt. 39.

27 For the divergence of the Peripatetic Commentators from Aristotle's views see S.Sambursky, 'Philoponus' interpretation of Aristotle's theory of light,' *Osiris,* XIII (1958), 114–126.

28 Alexander of Aphrodisias, *De Anima Libri Mantissa,* trans. Robert J. Richards, *Journal of the History of Behavioural Sciences,* 381; also, Sambursky, *ibid,* 116.

29 Philoponus, *De Anima,* quoted in Sambursky, 117–118 and discussed, 118–126. The idea that Philoponus 'completely rejects' Aristotle by changing light from a static to a kinetic phenomenon, and altering the meaning of Aristotle's 'energeia' is not accepted by R.Sorabji, see 'Directionality of Light,' in *John Philoponus, The Rejection of Aristotelian Science,* ed. R. Sorabji, i (Duckworth, 1986), 26–30.

30 The idea of light as an 'activity' of the luminous body in 'an outward direction' appears also in the *Enneads,* (IV, 5.7) of Plotinus (d. 270), see Sambursky, 116.

31 For the redefinition of 'light' in relation to the Atomists discussions of the divisibility of space and indivisibility of time, see Sorabji, *Time, Creation, and the Continuum,* 52–62, 384–90.

32 For the relationship of the pupillary picture and the possible etymological derivation of the term 'pupil', see R.Siegal, *Galen on Sense Perception,* 49–50; also, Galen 'On Anatomical Procedures,' *The Later Books,* trans. W.I.H. Duckworth (Cambridge, 1962), x, 3, 40; see note 79 below.

33 For the Stoic notion of 'presentation' of a coherent copy, see Hahm, 'Early Hellenistic Theories,' 88.

34 A.I.Sabra, 'Psychology versus mathematics: Ptolemy and Alhazen on the moon illusion,' in *Mathematics and its Applications to Science and Natural Philosophy in the Middle Ages,* eds. E.Grant and J.E.Murdoch (Cambridge University Press, 1987), 217–247.

35 Nicholas Pastore, *Selective History of Theories of Visual Perception: 1660–1950* (New York, 1971), 4–6.

36 See Proclus' 'Commentary on Euclid's Elements I' in *A Source Book in Greek Science,* 3–4.

37 Galen compares the light being cut off to the nerve that is severed and therefore loses all sensation, see, *De placitis Hippocratis et Platonis,* VII, 5.5–13, trans. De Lacy.

38 See Sorabji, 'John Philoponus,' in *John Philoponus,* 11–40 for the influential impetus theory as one strand in a broader attack on Aristotelian science.

39 We do not as yet have critical and comparative studies of the Hellenistic and Peripatetic sources of Islamic arguments in vision. For the relationship of the Greek and Islamic theories on the basis of the mathematical, physical, and medical criteria, see Lindberg, *Theories of Vision,* 18–58. He does not, however, cover the Aristotelian commentators.

40 For the overt expression of such aims and their application in specific treatises, see *al-*Kindī, 'Fī al-falsafat al-ūlā,' ed. ʿAbd al-Hādī Abū Rīda in *Rasāʾil* al-*Kindī al-falsafiyya,* I, 103 and ʿFī al-shuʿāʿātʾ (The Burning Mirrors), quoted by J.Jolivet and Roshdi Rashed, 'Al-Kindī, Abū Yūsuf Yaʿqūb Ibn Isḥāq al-sabbāḥ in *Dictionary of Scientific Biograph*y, 264; al-Rāzī, 'Al-Shukūk 'alā Jālīnūs' (Doubts Concerning Galen) in Shlomo Pines, 'Rāzī Critique de Galien,' rep. in *Studies in Arabic Versions of Greek Texts and in Mediaeval Science* (Leiden, 1986), 256–258; and Ibn al-Haytham, 'Al-Shukūk ʿalā Baṭlamyūs (Doubts Concerning Ptolemy), 162v quoted by Sh. Pines, 'Ibn al-Haytham's Critique of Ptolemy,' 547–48; optical

parts reproduced by A.I. Sabra, 'Ibn al-Haytham's Criticisms of Ptolemy's *Optics'* in *Journal of the History of Philosophy,* 145–149. Ibn al-Haytham's Kitāb *al-Manāẓir* (Optics), represents the culmination of this critical approach which is directly expressed Fātiḥ MS 3212, 4r) and applied as a programme of research.

41 B.Musallam, 'Avicenna between Aristotle and Galen,' in *Encyclopaedia Iranica,* ed. Ehsan Yarshater, III, fasc. 1 (London, 1986/87), 94–99. Bruce Eastwood, 'Al-Fārābī on Extramission, Intromission, and the Use of Platonic Visual Theory,' *Isis,* vol. 70 (1979), 422–425; Franz Rosenthal, 'On the Knowledge of Plato's Philosophy in the Islamic World,' *Islamic Culture,* 14 (1940), 386–422; specifically on vision, 412–16.

42 For the evaluation of Islamic developments, there is an acute need to secure a clear and firm ground in the preceding Greek arguments in vision and in related areas in late antiquity. Such a need has been underscored in a different context by R.Sorabji, ' Atoms and Divisible Leaps in Islamic Thought', in *Time, Creation and the Continuum* (Theories in Antiquity and the Early Middle Ages), chap. 25 (London, 1983), 384. In discussing atomism, he states that the Greek parallels (where they can be matched argument by argument) may be able to help in reconstructing the Islamic ones and that sometimes 'they actually put them in a new light and suggest a fresh meaning' particularly in the early period of Islamic thought.

43 For al-Kindī, see Jean Jolivet and Roshdi Rashed, *op. cit*. 261–67 which includes a detailed bibliography. al-Kindī's optics exists in a Latin translation from the Arabic, ed. by A.A.Björnbo and S.Vogl, 'Al-Kindi, Tideus und Pseudo-Euklid: Drei optische Werke,' in *Abhandlungen zur Geschichte der mathematischen Wissenschaften,* 26, no. 3 (1912), 3–41.

44 For a detailed reconstruction and study of al-Kindī's arguments see, D.C. Lindberg, 'Alkindi's Critique of Euclid's Theory of Vision,' *Isis,* 62 (1971), 469–89; reproduced in *Theories of Vision,* ii, 18–32, and a shorter version The Intromission-Extramission Controversy in Islamic Visual Theory: Alkindi versus Avicenna' in *Studies in Perception,* 137–160.

45 *De aspectibus,* Prop. 7 in Lindberg, *Theories,* 23; for the source of this argument as well as others in Theon of Alexandria's 'preface' to Euclid's *Optica,* see 20, 22; also Lindberg, 'Alkindi's Critique of Euclid', 476 (nt. 27), 477.

46 For the recognition of this problem by Ibn al-Haytham, see G.C.Hat field, and W.Epstein, The Sensory Core and the Medieval Foundations of Early Modern Perceptual Theory,' *Isis,* 70, 253 (1979), 368.

47 Prop. 9 in Lindberg, *Theories,* 22.

48 Prop. 14, *ibid,* 26–28.

49 Prop. 11, *ibid,* 24–25. Lindberg argues that for al-Kindī rays are not substantial entities but 'the impression of luminous bodies in dark bodies'.

50 Prop. 1–3, *ibid,* 20; Lindberg, 'Alkindi's Critique,' 474–75.

51 Prop. 13, *ibid,* 28–30.

52 A cone of continuous radiation has its origin in Ptolemy's *Optica*. For the difference between al-Kindī's and the Euclidean-Ptolemaic cones, see figure 37 in Lindberg, *Theories,* 226. The Arabic translation of Ptolemy's *Optica* was made from a manuscript lacking in Book 1 (on the general theory of vision) and the end of Book V dealing with refraction.

53 Prop. 10, Lindberg, *Theories,* 22. This also derives from Theon of Alexandria.

54 *The Book of the Ten Treatises on the Eye Ascribed to Ḥunayn ibn Isḥāq* (809–877) ed. and trans. Max Meyerhof (Cairo, 1928). Ḥunayn's version derives from a number of Galen's works which include *De usu partium* and *De placitis Hippocratis et Platonis.* For Arabic translations of Galen, see G. Bergstrasser *Hunayn ibn Ishāq und seine Schule* (Leiden, 1931), 15–24; M. Meyerhof, 'New Light on Hunain Ibn Ishāq and his Period,' *Isis,* 8 (1926), 45–685–724; G.Bergstrasser, *Neue Materialen zu Ḥunain ibn Ishāq's Galen Bibliographie* (Neudeln, Lichtenstein; 1966), 95–98.

55 For Hunayn's theory of vision see Lindberg, *Theories,* 33–42; for an analysis of some of the differences between Ḥunayn and Galen, see B.S.Eastwood, The *elements of Vision: The Micro-Cosmology of Galenic Visual Theory According to Hunain ibn Isḥāq* vol 72, part 5 (The American Philosophical Society, 1982), 1–59. For the origin and nature of the pneuma, see M.Ullmann, *Islamic Medicine* (Edinburgh, 1978), particularly 62–63 which is based on ʿAlī ibn al-ʿAbbās al-Majūsī's (d. c. 982–995; Latin Haly Abbas) classic medical encyclopedia, *Kitāb Kāmil al-Ṣināʿa al-Ṭibbiyya* or Kitāb al-Malakī (2 vols. Bulaq and Cairo, 1294/1877). Also, G.A.Russell, 'The anatomy of the eye in ʿAlī ibn al-ʿAbbās al-Majūsī: A textbook case', in C.Burnett and D.Jacquart (eds), *Constantine the African and ʿAlī ibn al-ʿAbbās al-Majūsī:* The Pantegni *and Related works* (Leiden, 1994), pp. 247–65.

56 *Ten Treatises,* Meyerhof, fol. 108.19–111.29, esp. 108.19–110.6, trans. 35–39.

57 *Ibid,* fol. 109.1–110.6, trans. 36–37.

58 For the influence of al-Kindī's *De aspectibus,* see Eilhard Wiedemann, 'Ueber das Leben von Ibn al Haiṯam und al Kindī,' *Jahrbuch für Photographic und Reproducktionstechnik,* 25 (1911) 6–7; Max Meyerhof, 'Die Optik der Araber,' *Zeitschrift für ophthalmalogische Optik* 8 (1920), 20. For Ḥunayn ibn Isḥāq, see J.Hirschberg, J.Lippert, E.Mittwoch, *Die arabischen Lehrbucher der Augenheilkunde,* (Berlin, 1905), 19–20; Max Meyerhof, 'Eine unbekannte arabische Augenheilkunde des 11. Jahrhunderts n. Chr.,' *Sudhoffs Archiv,* 20 (1928), 66–67. For his role in the transmission of Greek medical knowledge into Arabic, see G.Anawati and A.Z.Iskandar, 'Ḥunayn ibn Isḥāq', Dictionary *of Scientific Biography,* Suppl. I, 230–249.

59 The following discussion has been based on the section of the 'Doubts on Galen' (*Kitāb fī al-Shukūk ʿalā* Minus: Millī Malik, MS 4554/23, Tehran), reproduced in A.Z.Iskandar, 'Critical Studies in the Works of al-Rāzī and Ibn Sīnā' in *Proc. of the First Internal. Conference on Islamic Medicine,* II (Kuwait, 1981), 149–150.

60 For examples of self-operating valves and floats in hydraulic control, contemporary with al-Rāzī, see Banū Mūsa, *The Book of Ingenious Devices (Kitāb al-Ḥiyal),* trans, and annotated by D.Hill (Reidel, 1979).

61 Compare with Galen's explanation, *De usu partium,* trans. May, 476; *De placitis,* VII, 4.15. Also, S.Pines, 'Rāzī Critique de Galien,' in *Actes du VII*e *Congrès International d'Histoire des Sciences* (Jerusalem, 1953 (Paris n.d.), 480–87; reprinted in *The Collected Works of Shlomo Pines, Studies in Arabic Versions of*

Greek Texts and in Mediaeval Science, II (Jerusalem, 1986), 256–63. Pines suggests that al-Rāzī differed from Galen concerning the 'hollowness' of the optic nerve and the 'conduction' of the form of the visible object by the pneuma. For the Atomism of Rāzī in relation to Democritus, see Pines, *Beiträge zur Islamischen Atomenlehre* (Berlin, 1936).

62 Al-Rāzī, *Kitāb al-Manṣūr*, Book I, chap. 8 in *Trois traités d'anatomie arabes par Muḥammed ibn Zakariyyā' al-Rāzī, 'Alī ibn al-'Abbās et 'Alī ibn Sīnā,* ed. and trans. P.de Koning (Leiden, 1903), 53.

63 Al-Rāzī was acquainted with *De anima,* II attributed to Alexander of Aphrodisias, see Pines, 'Critique,' nt. 7, 487.

64 The *Kitāb al-Nafs,* section vi of the 'Ṭabī'iyāt' of the *Shifā*', ed. G.C.Anawati and Z.S.Zayed (Cairo, 1970); also *Avicenna's De Anima: Being the Psychological Part of Kitāb al-Shifā'* ed. F.Rahman (London, 1970), 115, 20–150, 19; *Avicenna's Psychology: An English Translation of the Kitāb al-Najāt,* trans. F.Rahman (Oxford, 1952), Book II, VI, ii.

65 For Ibn Sīnā's arguments see, D.C.Lindberg, 'The Intromission-Extramission Controversy in Islamic Visual Theory: Al-Kindi versus Avicenna' in *Studies in Perception;* also, *Theories of Vision,* 43–52.

66 *Najāt,* ii, 27:23–29; *Shifā*', 115:20—150:19. Although Ibn Sīnā has been labelled as 'Aristotelian' (Lindberg, *Theories,* 43–52), his approach in vision has yet to be studied in relationship to Aristotle and to Aristotelian commentators such as Themistius, Philoponus, and others. For the origin of some of Avicenna's arguments in Aristotle, and Alexander of Aphrodisias, see Rahman, *Kitāb al-Najāt,* 76–77.

67 *Najāt,* 29:3–15; *Dānishnāme,* trans. Mohammad Achena and H.Massé (Paris, 1955–58), 2:61; Lindberg, *Theories,* Fig. 6, 50. For a similar argument by Alexander of Aphrodisias, see above, note. 28.

68 *Najāt,* ii, 27:20; 29:31. For a textual comparison of the anatomy of the eye in *al-Qānūn,* with Galen see de Koning, *Trois Traités,* 660–66, and notes M through O, 799–802.

69 See G.A.Russell, *The Rusty Mirror of the Mind: Ibn Ṭufayl's Ḥayy ibn Yaqẓān and Ibn Sīnā's Psychology* (Philadelphia: The American Philosophical Society, 1966).

70 A.I.Sabra, *Theories of Light* (1967), 72, note 13.

71 The microfilms of manuscripts of the *Kitāb al-Manāẓir*, (Ahmet III 1819 and Fātiḥ 3212–16) have been provided by courtesy of the Topkapi and Süleymaniye libraries. The extant manuscripts of the first three of the seven books of Ibn al-Haytham's *Optics* have been meticulously edited by A.I.Sabra, see *Kitāb al-Manāẓir*: Books I–III *(On Direct Vision)* (Kuwait, 1983); also, for extensive bibliography, 'Ibn al-Haytham, Abū 'Alī al-Ḥasan ibn al-Ḥasan, in *The Dictionary of Scientific Biography,* VI, 189–210. [The following works were published after the manuscript of the present study first went into press: A.I.Sabra's *The Optics of Ibn al-Haytham. Books I–III* (London: The Warburg Institute, 1989), Vols I (Translation) and II (Introduction and Commentary). These two volumes represent a meticulous and exhaustive scholarship and should be consulted on all aspects of Ibn al-Haytham. Also Sabra, 'Form in Ibn al-Haytham's Theory of

Vision', *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften,* 5 (1989), pp. 115–40.]

72 The complex problem of 'colour' is outside the scope of this paper. For Ibn al-Haytham colour is a property of visible objects (whether they are opaque or transparent). For an analysis of the difference in Ibn al-Haytham's treatment of light and colour, see Rashed, 'Lumière et vision: L'application des mathématiques dans l'optique d'Ibn al-Haytham' in *Roemer et la vitesse de la lumière* 2 (Paris, 1978), 19–44, esp. 34–35.

73 *Kitāb al-Manāẓir,* I, iii, MS Fātiḥ 3212, 14v–15r. His approach is illustrated by a series of numerous experiments in chapters ii 'The Properties of Lights' and iii, 'The Properties of Lights and the Manner of their Radiation'. For Ibn al-Haytham's physical optics, see R.Rashed, 'Optique géométrique et doctrine optique chez Ibn al-Haytham', *Archive for the History of Exact Sciences,* 6 (1970), pp. 271–98, esp. 274–6; and A.I.Sabra, 'The Physical and the Mathematical in Ibn al-Haytham's Theory of Light and Vision', in *The Commemoration Volume of al-Bīrūnī International Conference in Tehran* (Tehran, 1976), pp. 439–78, 457–9.

74 *K. al-Manāẓir,* I, ii, MS Fātiḥ 3212, 5v–8r. For a further experimental demonstration of this principle using a specially constructed sighting instrument with two variable slits (vertical and horizontal) in his 'Treatise on the Light of the Moon,' see M.Schramm, *Ibn al-Haytham's Weg zur Physik* (Wiesbaden, 1962), 164–200.

75 *K. al-Manāẓir,* I, iii, MS Fātiḥ 3212; for a clear statement, see esp. 23v–26r.

76 *K. al-Manāẓir,* I, iii, MS Fātiḥ 3212, 22r–25r.

77 The principles based on the experimental investigations in his *Optics* are described in Ibn al-Haytham's 'Discourse on Light' *(Fī al-Ḍaw'),* critical trans, by R.Rashed, 'Le *Discours de la Lumière* d'Ibn al-Haytham (Alhazen)', *Revue d'histoire des sciences et de leurs applications,* 21 (1968), 198–224.

78 For a discussion of Ibn al-Haytham's experimental results for refraction, see Sabra, 'Ibn al-Haytham', in *Dictionary of Scientific Biography,* 194; Sabra, 'Explanation of Reflection and Refraction: Ibn al-Haytham, Descartes, Newton', in *Actes de dixième Congrès international d'histoire des sciences, Ithaca, 1962* (Paris, 1964), I, 551–54; and Rashed, 'Lumière et Vision', 19–44 esp. 30–44; and 'Optique géométrique,' 293–6.

79 For 'images' in mirrors and the eye, see Galen, *On Anatomical Procedures,* X, 3, 40, transl. Duckworth; also *De usu partium,* X, 6, 479, transl. May. Galen's localization of the pupillary picture on the anterior surface of the crystalline (on the arachnoid layer) was continued by Ḥunayn ibn Isḥāq, *The Ten Treatises,* 109, trans. Meyerhof, 36–37. For the expression of the same concept by the proponents of intromission, see Ayyūb al-Ruhāwī's (Job of Edessa: d. after 832) *Book of Treasures,* trans. A.Mingana, Disc. III, chap. iv (Cambridge, 1935), 134. He argued that just as sunlight is cast on a wall from polished brass or silver vessels or water, 'in this same way, when light of the sun shines on the eye, it causes a reflection in it of the outside objects or forms'; Ibn Sīnā, 'On the Soul,' *Najāt,* ii, 27, fol. 30; also in *Le Livre de Science, (The Dānishnāme),* trans. Mohammad Achena and Henri Massé, II (Paris, 1955–58), 60; and *A Compendium on the Soul,* trans. Edward A.van Dyck (Verona, 1906), 51–52; Lindberg, *Theories of Vision,* 49.

80 The association of vision with the appearance of a 'copy' in the pupil goes back to Democritus, see Crombie, *The Mechanistic Hypothesis,* 6, nt. 9; Lindberg, *Theories of Vision,* 3.

81 For the earliest occurrence of *al-bayt al-muzlim* in a ninth-century Arabic source, deriving from Greek works on 'burning mirrors, see Sabra, 'Ibn al-Haytham,' 204, nt. 19. That light passing through an aperture casts an image of its source was recognized and described, for example, in the Pseudo-Aristotelian *Problemata* and the Pseudo-Euclidean *De Speculis,* an Islamic compilation; see Lindberg, 'The Theory of Pinhole Images from Antiquity to the Thirteenth Century,' *Archive for History of Exact Sciences,* 5 (1968), 154–176. Here the term 'pinhole' is used in a broader sense of image-forming aperture of various sizes and shapes.

82 For Ibn al-Haytham's arrival at the concept of a ray or 'the smallest of lights' in relation to aperture, see Sabra, 'Ibn al-Haytham,' *Dictionary of Scientific Biography,* 191–2.

83 *K. al-Manāẓir*, I, vi, MS Fātiḥ 3212, 115r 9–115v 6.

84 *K. al-Manāẓir*, vi, MS Fātiḥ 3212, 115v 7–116r 4.

85 *K. al-Manāẓir*, I, vi, MS Fātiḥ 3212, 116r 4–116v 13.

86 In the treatise, 'On the Shape of the Eclipse,' which was composed after the *K. al-Manāẓir*, Ibn al-Haytham shows a clear understanding of the principles of a pinhole camera and the projection of a distinct image, taking into account the diameter of the aperture as well as the distance of the screen and the projected object from it. This treatise has been the subject of a number of studies, see Sabra, 'Ibn al-Haytham,' 195–6; see particularly M.Schramm, 'Die Camera obscura effektes,' *Weg zur Physik,* 202–74.

87 Ibn al-Haytham's 'descriptive' anatomy is given in chap. v and 'functional' anatomy in chap. vii of the *Kitāb al-Manāẓir*.

88 Muṣṭafā Naẓīf, in his comprehensive two-volume study of Ibn al-Haytham's optical researches, drew attention to Ibn al-Haytham's detailed description of the eye. See, *al-Ḥasan ibn al Haytham buḥūthuthū wa kushūfuhu al-baṣariyya* (Arabic text), vol. I (Cairo, 1942–43), sections 48–49, 205–217. His interpretation of the anatomy, represented by his diagram of Ibn al-Haytham's eye (211), which has been taken as the standard reference, is erroneous.

89 For the correct interpretation of Ibn al-Haytham's descriptive anatomy it is important to point out that he uses the same term to denote several structures. For example, *al-multaḥima* refers to the conjunctiva, orbital fat (which has been mistaken as a layer in modern interpretations) as well as the sclera (for which he sometimes uses *bayāḍ al-multaḥima*). In each case the specific usage or application of the label can be determined without any ambiguity from the exact and detailed description and the context.

90 What is known of Ibn al-Haytham's acquaintance with Galen's texts (and his lost summaries of Galen) derives from the medical historiography of Ibn Abī Uṣaybiʿa (1203–1270), *ʿUyūn al-Anbāʾ fī ṭabaqāt al-aṭibbāʾ*,, A.Müller, ed., vol. II (Cairo-Konigsberg, 1882–84), 90–98. He had access to an autograph of Ibn

al-Haytham's autobiography, and a list of Ibn al-Haytham's works; see G. Nebbia, 'Ibn al-Haytham nel millesimo anniversario della nascita,' *Physis,* IX, 2 (1967), 179–180 where under medicine thirty titles are listed. For the relationship of Ibn al-Haytham's autobiography to Galen's *De libris propriis* and to 'De methodo medendi' (which were available in Arabic in Ḥunayn ibn Isḥāq's translation), see F.Rosenthal, 'Die arabische Autobiographic', Studio *arabica,* I, *Analecta Orientalia,* No. 14 (Rome, 1937), 7–8; G.Strohmaier, 'Galen in Arabic: Prospects and Projects,' in *Galen: Problems and Prospects,* ed. V.Nutton (London, 1981), 187–196; also, M.Schramm, 'Zur Entwicklung der physiologischen Optik der arabischen Literatur', *Sudhoff Archiv für Geschichte der Medizin und der Naturwissenschaften,* 43 (1959), 289–92, esp. 292. For the evaluation of sources and extensive bibliography, see Sabra, 'Ibn al-Haytham', in *Dictionary of Scientific Biography,* 189–190, 209; Sabra, 'Introduction', *The Optics of Ibn al-Haytham,* II.

91 *Kitāb al-Manāẓir,* I, v, MS Fātiḥ MS 3212, 73v 5.

92 Here again the same term *(al-'inabiyya)* is used to denote several structures: the iris, the uveal coat (that is, the ciliary body and the choroid as an extension of the inner sheath of the optic nerve), and the uveal chamber, which is a combination, in modern terms, of the posterior and vitreal chambers. This is not the Galenic usage where the uvea, or the 'grape-like', refers only to the iris and the ciliary body and not to the whole choroid coat. See trans. May, *De usu partium,* 475. Ibn al-Haytham uses the term 'funnel' (Arabic *qimā'*) to describe the expansion of the optic nerve. Arabic funnels were both conical as well as round in shape as found in treatises on mechanical devices by the Banū Mūsā (10th c.) and al-Jazarī (12th c.). I owe confirmation of this to Donald Hill.

93 Ibn al-Haytham's description of the anterior and the posterior chambers of the eye has also been missed. The Naẓīf diagram or its reproductions show no anterior chamber between the cornea and the iris.

94 The term 'lens' is used here simply as a label for the structure without identifying it with the modern conception of a point-focus function which has no place in Ibn al-Haytham's usage.

95 *K. al-Manāẓir,* I, v, MS Fātiḥ 3212, 73v–74r; vii, 130v 6–12.

96 *K. al-Manāẓir,* I, MS Fātiḥ 3212, v, 74r–74v; 130r 10–13.

97 *K. al-Manāẓir,* I, vii, Fātiḥ 3212, 130v 11–131v 1–5. The arachnoid (al-'ankabūtiyya in Ḥunayn ibn Isḥāq and subsequently in descriptions of ocular anatomy, as in 'Alī ibn 'Īsā' (see below, nt. 99), was regarded as a thin membrane covering the anterior part of the crystalline humour. In Ibn al-Haytham it is used differently. Preserving the combined spherical shape of the lens and the vitreous, it constitutes the innermost layer in the back of the eye. On the basis of his consideration of the *layers* maintaining the spherical shape of the *fluid* parts of the eye, it can be interpreted to be continuous with the retina. There is, however, no reference to the retina *(al-shabakiyya)* or to the choroid coat *(al-mashīmiyya)* in Ibn al-Haytham. As the retina was part of the Galenic description of ocular anatomy, one can only conclude that its omission was not an arbitrary oversight, but part of a deliberate disregard of all that was irrelevant to his functional analysis. Similarly, there is no concern with the number of layers or fluids in the eye, nor with any notion of 'nourishment' of the lens as in traditional descriptions.

98 For comparison, see Galen, *De usu partium,* X, 463–503, trans. May; and for the description of the nerves in relation to the brain, *De placitis,* VII, 3–8, trans. de Lacy; for the 'eye', esp. VII, 5.22–30, trans. de Lacy.

99 This difference is clearly illustrated by the definition of the lens as 'ice-like' *(al-jalīdiyya)*. In Ibn al-Haytham, the reference is to the nature of its transparency, which is partly dense *(ghaliz)* and partly clear *(shafīf);* whereas in ʿAlī ibn ʿĪsā the reference is to its 'cold' and 'dry' nature. J.Hirschberg, and J.Lippert, ʿAlī ibn ʿĪsā (Leipzig, 1904), 8–10; C.E.Wood, Tadhkirat of *ʿAlī ibn ʿĪsā, Memorandum Book of a Tenth Century Oculist for the Use of Modern Ophthalmologists,* trans. Casey A.Wood, Bk I, chap. 20 (Chicago, 1936). The former is a description of observable characteristics, the latter is a qualitative account on the basis of a theoretical doctrine which is already indicated by the title of ʿAlī ibn ʿĪsā's chapter, 'Of the Nature of the Eye and Its Temperaments.' It is from this approach that Ibn al-Haytham sharply deviates.

100 Prior to Ibn al-Haytham, there is no indication that the spatial relationship of the structures of the eye was considered at all. As Schramm ('Physiol. Optik', 290) rightly points out, this was missing in Galen's description in spite of its great detail.

101 The traditional descriptions of the lens account for its 'flattened' spherical shape with reference to its being less liable to receive injury and having greater surface area for contact with the impressions of objects conveyed by the pneuma. See Galen, *De usu partium,* X, 6, 15, trans. May; Ḥunayn, *Ten Treatises,* 3–4. For Galen's consideration of the lens in geometrical terms, see Max Simon, *Sieben Bucher Anatomie des Galen,* Bk. II (Leipzig, 1906), 35–36; Schramm, 'Physiol. Optik', 199, nt. 1 and 200, nt. 1.

102 *K. al-Manāẓir,* I, v, MS Fātiḥ 3212, 74r. 4–7; vii, 130r.

103 *K. al-Manāẓir,* I, v, MS Fātiḥ 3212, 74r. 10–13; 75v, 6–10; vii, 130r.

104 *K. al-Manāẓir,* I, v, MS Fātiḥ 3212, 76r. 8–13; 76v. 8–10.

105 *K. al-Manāẓir,* I, v, MS Fātiḥ 3212, 75v.–76r. 7. Ibn al-Haytham's emphasis on the external surface of the cornea being part of the globe of the eye is in the sense of a continuation of the sclera, and not in terms of having the same radial centre.

106 *K. al-Manāẓir,* I, v, MS Fātiḥ 3212, 76v 8–10; 78v. 8–13.

107 *K. al-Manāẓir,* I, v, MS Fātiḥ 3212, 76v. 5–78v.

108 See below, note 125.

109 *K. al-Manāẓir,* I, v, MS Fātiḥ 3212, 76r. 6–10; 78v (esp. 8–14)–80v.

110 This is exemplified by the diagram inserted into the printed edition of the Latin translation of Ibn al-Haytham's *K. al-Manāẓir*. See *De Aspectibus in Opticae thesaurus Alhazeni Arabis libri septem,* ed. F.Risner (Basel, 1572).

111 Lindberg, *Theories,* 69.

112 *K. al-Manāẓir,* I, vi, MS Fātiḥ 3212, 115v. 7–116r. 4.

113 *K. al-Manāẓir,* I, vi, MS Fātiḥ 3212, 97r.; II, ii, 7v.

114 For a discussion of Ibn al-Haytham's use of mechanical analogies for refraction, see A.I.Sabra, 'Explanation of Optical Reflection and Refraction: Ibn al-Haytham, Descartes, Newton', 551–54; also. R.Rashed, 'Lumière et Vision', 28–32, 44.

115 *K. al-Manāẓir,* I, vi, Fātiḥ MS 3212, 97v–98r, 100v–105r. For the full translation of this passage, see Sabra, 'Ibn al-Haytham and the Visual Ray Hypothesis' in *The Ismā'īlī Contributions to Islamic Culture,* ed. S.H.Nasr (Tehran, 1977), 193–205.

116 *K. al-Manāẓir,* I, iv, MS Fātiḥ 3212, 90v.

117 *K. al-Manāẓir,* I, iv, MS Fātiḥ 3212, 67r.; vi, 107r.–108r.

118 *K. al-Manāẓir,* I, vi, MS Fātiḥ 3212, 106v–107r, 117r–118r.

119 *K. al-Manāẓir,* I, vii, MS Fātiḥ 3212, 130r.

120 Sabra, 'Ibn al-Haytham', 193–194; R.Rashed, 'Lumière et Vision', 40–41.

121 *K. al-Manāẓir,* I, vi, MS Fātiḥ 3212, 112r–113r.

122 *K. al-Manāẓir,* I, ii, MS Fātiḥ 3213, 44v–45r.

123 *K. al-Mānāẓir* I, vi, MS Fātiḥ 3212, 108v–114r.

124 On the evidence of convergent eye movements, Ibn al-Haytham shows the weaknesses of Ptolemy's argument which is based on the central or axial ray of the visual cone. For the passage in question from Ibn al-Haytham's Treatise 'Doubts About Ptolemy,' see A.I.Sabra, 'Ibn al-Haytham's Criticisms of Ptolemy's Optics', *Journal of the History of Philosophy,* IV, No. 2 (April, 1966), 145–149; esp. 147 and 148.

125 Schramm, *Weg zur Physik,* 234.

126 For a study of Ibn al-Haytham's approach to percepcion, see A.I.Sabra, 'Sensation and Inference in Alhazen's Theory of Visual Perception' in *Perception,* 169–185; also Sabra, 'Form in Ibn al-Haytham's Theory of Vision', 130–40.

127 For the optics of Kamāl al-Dīn, see E.Wiedemann, 'Zu Ibn al-Haitham's *Optik', Archiv für die Geschichte der Naturwissenschaften und der Technik,* 3 (1910), pp. 1–53. For further bibliography, see R.Rashed, 'Kamāl al-Dīn al-Fārisī' in *Dictionary of Scientific Biography,* pp. 212–9.

128 M.Schramm, 'Physiol. Optik', 299–316.

21
The Western reception of Arabic optics

DAVID C.LINDBERG

One of the most interesting and significant features of the history of early optics is its continuity across cultural and linguistic boundaries. This is not, of course, to suggest that the science of optics remained totally static, escaping all need to adapt to changing cultural, linguistic and philosophical circumstances. But it is important to understand that despite development and adaptation, optics retained a recognizable identity from the ancient Greeks to the beginning of the seventeenth century.

This continuity is particularly striking from Ibn al-Haytham in the eleventh century to Johannes Kepler in the seventeenth. Although it is indisputable that optical theory saw interesting and important developments during this period, there was surprisingly little change in the questions to which it was addressed, the basic assumptions on which it was founded and the criteria of theoretical success that it had to meet. Central, therefore, to the history of early optics is the problem of transmission and assimilation. This chapter will be devoted to the Latin reception of Arabic optics.

THE TRANSLATIONS

Before the translations of the twelfth and thirteenth centuries, the West had access to only the most meager optical fare. In the encyclopaedias of Pliny the Elder (d. AD 79), Solinus (fl. third or fourth century) and Isidore of Seville (seventh century), one finds elementary discussions of various optical phenomena, but optical theory only at the most rudimentary level. We learn, for example, that sight occurs by light issuing from the eye, that the seat of vision is the pupil or centre of the eye, that light moves more swiftly than sound, that Tiberius Caesar could see in the dark and that a rainbow is caused by solar light incident on a hollow cloud. We also learn some elementary ocular anatomy. Apart from Pliny's brief account of the shape of shadows as a function of the relative diameters of the luminous and shadow-casting bodies, mathematical analysis is entirely absent.[1]

For philosophically more sophisticated discussions that fit light and sight into a larger theoretical framework and offer an appraisal of alternatives, we must abandon the encyclopedias and turn to other genres of literature. In a variety of theological works, including his *The Literal Meaning of Genesis,* Augustine of Hippo (354–430) drew on the light metaphysics of the Neoplatonic tradition to explain the creation of the world, the relationship between body and soul and the acquisition of knowledge. Augustine also dealt briefly but cogently with the nature of visible light and the process of visual perception. Another source, available from the fourth century but not prominent until the twelfth, was the first half of Plato's *Timaeus,* which offered a fairly coherent account of the nature of light and the manner in which motions were transmitted from a visible object to the soul of the observer to produce visual perception.

Several features of this early Latin optical literature require emphasis. First, there were no treatises devoted exclusively to optical subjects; obviously, optics was not yet conceived as a substantial, autonomous discipline, requiring its own specialized literature; rather, it was a piece of general knowledge, relevant to a variety of other concerns and therefore deserving modest attention (at most) in physical, metaphysical, theological and encyclopedic works. Second, such optical discussions as one finds were almost never mathematical in character. The questions at issue were the *nature* of light and the *nature* of visual perception, rather than the mathematics of propagation and perspective. Third, light was generally conceived as a corporeal entity, perhaps affiliated with fire. And fourth, vision was generally thought to occur through a process of extramission—visual fire travelling from the eye to the visible object (and perhaps back again). Beyond such elementary matters, optics seldom ventured.

The translations of the twelfth and thirteenth century produced a dramatic transformation. For the first time, Western Christendom found itself in possession of entire treatises devoted to optics. Some of these were of Arabic origin; others were Greek treatises, transmitted through Arabic mediation.[2]

The earliest translated treatise devoted entirely to optical matters was the *De oculis* of Ḥunayn ibn Isḥāq, put into Latin by Constantine the African near the end of the eleventh century (and thereafter attributed either to Constantine or to Galen). This work offered a Galenic account of the anatomy and physiology of the eye and a defence of Galen's theory of vision. Other treatises touching upon ocular anatomy and physiology, as well as diseases of the eye, soon followed: the *Pantegni* of ʿAlī ibn al-ʿAbbās (translated by Constantine and again in the next century by Stephen of Antioch), Ibn Rushd's *Liber canonis,* al-Rāzī's *Liber ad Almansorem* and Yūḥannā ibn Sarābiyūn's *Practica* (the latter three all rendered by Gerard of Cremona in the second half of the twelfth century).

The twelfth century saw the translation of a series of optical treatises that were heavily, though not exclusively, mathematical. Three Greek treatises were among the earliest: the *Optica* and *Catoptrica* attributed to Euclid and the *Optica* attributed to Ptolemy, all rendered about the middle or early in the second half of the twelfth century. The Euclidean *Optica* appeared in at least three translations, two from the Arabic and one from the Greek, while the Ptolemaic *Optica* was rendered from an incomplete and defective Arabic version.[3] These were soon joined by a group of translations by Gerard of Cremona or his school: the *De aspectibus* of al-Kindī, the *De crepusculis* of Ibn Muʿādh, the *De speculis* or *De aspectibus* of Tideus, a *De speculis* (frequently attributed to Euclid) compiled in Islam from Greek sources and a work possibly translated by Gerard, the *De speculis comburentibus* of Ibn al-Haytham. The treatise that would, in the long run, prove the most influential, Ibn al-Haytham's *De aspectibus,* was translated (by an anonymous translator) late in the twelfth century or early in the thirteenth (Lindberg 1976:209–11).

There was a third and final category of literature that dealt with optical issues —namely, works on natural philosophy, especially meteorology and perception. Here the most influential treatises were Aristotle's *De anima, De sensu* and *Meteorologica* (the first and last of which were available in twelfth- or thirteenth-century translations from the Arabic); Ibn Sīnā's *De anima* (translated in the second half of the twelfth century); and Ibn Rushd's commentary on Aristotle's *De anima* and epitome of Aristotle's *Parva naturalia* (both translated, it appears, early in the thirteenth century) (Lindberg 1976:212–13).

Though incomplete, this list of optical literature makes clear the dramatic transformation in the quantity and quality of optical literature available in the West, owing to the acquisition of Greek and Arabic learning. During the early Middle Ages, Christendom was struggling to conserve scraps of the ancient heritage; after the translations, the struggle was to assimilate a vast and diverse body of new knowledge.

THE MATHEMATICS OF LIGHT AND VISION

One of the most striking features of the new optical literature was its mathematical form. Though certainly not characteristic of all of the new material, mathematization was nonetheless conspicuous. The propositional structure of some of the treatises and the geometrical form of much of the argument were unprecedented in the optical experience of the West. Euclid's *Optica* (entitled *De visu* or *De aspectibus* in its Latin translation) set the tone: from a set of postulates, the treatise proceeds to fifty-eight propositions, containing geometrical demonstrations accompanied by geometrical diagrams. Insofar as possible, Euclid reduces optics to the analysis of geometrical rays emanating (in straight lines unless reflected or refracted) from an observer's eye in the form of a cone. The cone of rays serves as the basis for a mathematical theory of perspective.[4]

The geometrical approach to light and vision was elaborated in other works: the *Catoptrica* (or *De speculis*) attributed to Euclid, Ptolemy's *Optica* (or *De aspectibus*), al-Kindī's *De aspectibus* and especially the *De speculis comburentibus* and monumental *De aspectibus* of Ibn al-Haytham. While none of these, except perhaps the two treatises on mirrors, can be regarded as purely mathematical in content, mathematics was prominent in all of them. No reader could overlook the geometrical argument; and even the most cursory inspection would reveal the geometrical diagrams.

The geometrical approach of these treatises was not merely novel and striking; it was also easy to assimilate. There were no obvious theological or philosophical objections and no significant cultural obstacles to the use of mathematics to analyze optical phenomena. Even those who had reservations in principle about the extent to which mathematics was applicable to nature could find no fault with the geometrical approach to optics—as long as it was not supposed that the geometrical approach was the only approach.[5] Greek and Arabic geometrical optics constituted a technical achievement that was simply too impressive to be overlooked or dismissed.

The first Western scholar to reveal the influence of the geometrical approach was Robert Grosseteste (*c.* 1168–1253), writing (probably) in the early 1230s.[6] Grosseteste, who had read Euclid and al-Kindī, was inspired to set forth a geometrical definition of *perspectiva* and to lay down a geometrical program for the analysis of radiating force. In his *De iride* he defined *perspectiva* as 'the science based on figures containing radiant lines and surfaces, whether that radiation is emitted by the sun, the stars, or some other radiant body' (Grant 1974:389). Grosseteste proceeded to identify the principal subdivisions of *perspectiva* according to the mode of propagation of light: rectilinear, reflected and refracted. In his *De lineis, angulis, et figuris,* Grosseteste issued a manifesto for the geometrization of nature through the geometrization of light and other forms of radiation:

> Now all causes of natural effects must be expressed by means of lines, angles, and figures, for otherwise it is impossible to grasp their explanation. This is evident as follows. A natural agent multiplies its power from itself to the recipient, whether it acts on sense or on matter. This power is sometimes called species, sometimes a likeness, and it is the same thing whatever it may be called; and the agent sends the same power into sense and into matter, or into its own contrary, as heat sends the same thing into the sense of touch and into a cold body.
>
> (Grant 1974:385)

Roger Bacon (*c.* 1220–*c.* 1292) was familiar with all of the sources available to Grosseteste, but he also knew Ptolemy's *Optica* and Ibn al-Haytham's *De aspectibus,* where the promise of the mathematical approach had been much more fully realized. The impact of these two treatises, especially that of Ibn al-Haytham, on the mathematical content and sophistication of Bacon's optical writings was dramatic.

Bacon presents a comprehensive account of the geometry of radiation, drawn principally from Ibn al-Haytham. Bacon defines five modes of propagation: rectilinear, reflected, refracted, accidental (by which he intends secondary radiation, originating at points in a beam of primary light) and the 'tortuous' mode characteristic of certain animated media.[7] He presents a comprehensive statement of the law of reflection, which not only affirms the equality of the angles of incidence and reflection but also defines the plane of the incident and reflected rays in relation to the mirror surface.[8] Bacon undertakes a careful account of the geometrical principles of refraction, defining the path of the refracted ray (in geometrical, although not numerical, terms) for various configurations of rare and dense media and for both plane and spherical transparent interfaces.[9] Still following his Greek and Arabic sources, Bacon locates the image of an object seen by either reflected or refracted radiation at the intersection of the incident ray (extended backward from the eye) and the perpendicular dropped from the object to the reflecting or refracting surface. Bacon proceeds to apply these principles to certain cases of special interests—the burning mirror and the burning glass.[10]

Significant as these assimilated optical principles were, perhaps the most important thing that Bacon learned from his sources was how to conceive radiation from an extended object. From al-Kindī and Ibn al-Haytham, Bacon gathered that light radiates independently, in all directions, from every point (or small region) of a visible object. This conception of a fundamentally incoherent process of radiation was foreign to Greek antiquity; first formulated by al-Kindī and subsequently applied by Ibn al-Haytham, it was to prove one of the fundamental principles of geometrical optics, playing a critical role in both theories of radiation and theories of vision.

Bacon could not match the mathematical sophistication of his best source, Ibn al-Haytham. None the less, what he communicated, he communicated faithfully and with considerable skill. Others, apparently inspired by Bacon's example, adopted a similar approach to optics (Lindberg 1971a). Witelo (d. after 1281) wrote an enormous *Perspectives,* an encyclopedia of optics, in which he attempted (with only an occasional blunder) to capture the entire Greek and Arabic optical achievement. And John Pecham (d. 1292), a younger Franciscan contemporary of Bacon, wrote a brief, popular textbook, entitled *Perspectiva communis,* which skilfully summarized the essentials of optics.[11] Through these sources, as well as the Greek and Arabic originals (which continued to circulate in Latin translation), Western scholars learned how to do optics the mathematical way.

THE NATURE OF LIGHT

When the geometry of radiation was introduced to the West, it had the advantage not only of novelty, but also of philosophical neutrality.[12] Moreover, it was a relatively unified body of doctrine, marred by few internal disagreements. By contrast, the nature of the radiating entity and the direction of its emission were hotly disputed questions, which impinged on other issues and therefore called for the prudent selection of scholarly opinion and the thoughtful weighing of scholarly argument.

Greek theories of light came in many varieties. According to the atomists light was a corporeal emanation. Vision occurred, in their opinion, when a film of atoms was transmitted from the visible object to the observer's eye, conveying the visible properties of the former to the soul atoms of the latter. More influential, in the long run, was Aristotle's belief that light is a state of the transparent medium whereby its transparency is fully activated, colour being a further qualitative change induced in the activated transparency by a coloured object; the latter qualitative change can be transmitted by the medium to the eye of an observer, who thereby sees. The Pythagoreans apparently developed a theory of visual fire emanating from the eye—a theory of which we find continuing echoes throughout antiquity and the Middle Ages. This theory of visual emanation was developed by Plato, utilized by Euclid and Ptolemy in their mathematical theories of vision, and transformed by Galen and the Stoics into a theory of visual pneuma (Lindberg 1976: ch. 1).

As if this were not complicated enough, in late antiquity Plotinus (d. 270), the founder of Neoplatonism, developed an emanationist metaphysic in which all being is derived from the 'One' by a process of emanation analogous to the radiation of light. In the physical world, as in the metaphysical, all things are centres of activity, projecting images of themselves into their surroundings. This radiating light is totally incorporeal, consisting neither of moving corpuscles (as the atomists maintained) nor of qualitative changes in a medium (as Aristotle urged); rather, incorporeal light leaps instantaneously over the medium without in any way interacting with it. Finally, Plotinus distinguished between radiating light and light in a luminous body, the latter functioning as the corporeal form of the luminous object (Lindberg 1986:9–12).

This complex heritage was transmitted to Islam, where it was taken up by a series of able philosophers. Al-Kindī (d. *c.* 873), one of the first Arabic philosophers, adopted the emanationist metaphysic of Plotinus, arguing that 'everything in the world, whether substance or accident, produces rays in the manner of stars,…so that every place in the world contains rays from everything that has actual existence' (d'Alverny and Hudry 1974:224, 228). However, when it came to the nature of the radiating entity, al-Kindī departed from Plotinus, insisting that light is an 'impression' made by the luminous body in a transparent medium (Lindberg 1986:12–14).

The other major Greek traditions also found defenders within the Islamic world. Ḥunayn ibn Isḥāq (d. *c.* 877), a Nestorian Christian instrumental in the translation of Greek learning into Arabic, adopted and promulgated the Stoic or Galenic theory, according to which visual spirit emerges from the eye and converts the air into a sensitive organ—an extension of the optic nerve, capable of perceiving objects with which it comes into contact (Eastwood 1982:31–46; Lindberg 1976:37–41). Ibn Sīnā (980–1037) accepted the Aristotelian position that light is a quality of the transparent medium induced by luminous objects. However, perhaps borrowing from the Neoplatonic tradition, Ibn Sīnā distinguished between light as it exists in luminous bodies and light in the medium (*lux* and *lumen,* respectively, in the Latin translation of his work); he also identified a third entity, the ray or radiance 'which appears around bodies…as though it were something emanating from them' (Avicenna, *Avicenna Latinus,* pp. 170–2).

Ibn al-Haytham (965–*c.* 1039), whose major contributions to optics were in the geometrical realm, made no sustained or systematic attempt to explore the nature of light. It is evident from his works, however, that he accepted the basic conception of the 'physicists', who (in his view) regarded light as an essential or accidental form, respectively, of self-luminous or illuminated bodies.[13] He thus offered the important distinction between essential and accidental or borrowed light. He also treated light in a transparent medium as a form, transmitted from the luminous or illuminated body to a recipient. And he maintained (with Ibn Sīnā and against Aristotle) that light, as well as colour, is an object of vision; the forms of light and colour are propagated together through a suitable medium and act simultaneously on the visual power (Lindberg 1978a:356–7).

Finally, the Spanish Muslim Ibn Rushd (d. 1198), while generally supporting Aristotle's theory of light and colour, endeavoured to deal with the perplexing phenomenon of different colours apparently occupying the same place without mixing or interfering (as when the forms of a white thing and a black thing simultaneously enter an observer's pupil). Ibn Rushd concluded that forms in the medium have neither spiritual nor corporeal existence, but possess a status intermediate between those two extremes (*Epitome,* pp. 15–16).

Our primary task in this chapter, of course, is not to recount the Arabic optical contribution, but to assess its influence on the West. Western scholars had access to the full range of Greek and Arabic opinion on the nature of light and from these opinions pieced together a variety of theories. There is no question that al-Kindī exerted a powerful influence with his conception of all things as centres of activity, radiating their power or image in all directions. This was easily coupled with Ibn Sīnā's distinction between *lux* and *lumen—lux* functioning as the active form of luminous bodies, from which issues *lumen,* the likeness or form in the medium. Perhaps the most systematic expression of these views was in the doctrine of the 'multiplication of species' developed by Grosseteste and Bacon, according to which species or likenesses emanate in all directions from everything to produce all natural effects (Lindberg 1986:14–23; Bacon, *Philosophy of Nature,* pp. xlix–lxxi).

What was perhaps most significant about Western theories of the nature of light was the unanimous repudiation of the Plotinian incorporealist position. Under the influence of Aristotle, al-Kindī, Ibn Sīnā and Ibn al-Haytham, virtually every Western scholar who discussed the nature of light took it to be a quality or modification of a corporeal medium. Platonizing followers of Grosseteste and Bacon joined strict Aristotelians in the conviction that light and the medium were inextricably related, so that if there is no medium, there can be no radiating light. Except for the anomalous William of Ockham, who was prepared to entertain and even defend action at a distance (without mediation of any kind), this position prevailed without opposition until the end of the fifteenth century, when Marsilio Ficino attempted to revive the theory of Plotinus (Lindberg 1986:14–29).

THEORIES OF VISION

Variety among theories of vision was no less bewildering than opinion on the nature of light. I have argued elsewhere that ancient theories of vision fall into three categories (Lindberg 1976:85–6; 1978a:341–2). The extramission theory of Euclid and Ptolemy, which conceived of visual radiation emanating from the eye, was fundamentally mathematical in its aims, offering (above all) a theory of visual perspective. The intromission theories of the atomists and Aristotle were primarily physical theories, designed to account for contact between observer and observed object and to explain the physics of transmission.[14] And, finally, the Galenic theory, while not devoid of mathematical and physical content, differed from its rivals in its inclusion of anatomical and physiological detail.

Each theory combined certain explanatory advantages with various explanatory deficiencies. The mathematical theory of Euclid offered a geometrical explanation of the perception of space through its visual cone, while virtually ignoring the physical question of contact between observer and observed; in Ptolemy's hands this theory acquired substantial physical content,[15] but its virtues and its influence remained primarily mathematical. The physical theory of Aristotle admirably resolved the problem of physical contact, but (in the form in which Aristotle presented it) was devoid of mathematical content or potentiality. The physical theory of the atomists may have fallen short even of a satisfactory physical account—Aristotle certainly thought so—but in any case displayed no mathematical aspirations. Galen's theory of visual pneuma, finally, was successful primarily as an account of the anatomy and physiology of vision; it offered little in the way of a theory of perspective, and its physical theory seems to have held little appeal for natural philosophers. Each theory suffered from limitations in scope. The choice of a theory of vision, therefore, was to a considerable degree the choice of which criteria—mathematical, physical or medical—one proposed to satisfy.[16]

A pair of brilliant theoretical insights in Islam transformed the debate. First, as I noted above, al-Kindī argued that radiation from a luminous body must be understood as an incoherent process, in which the body does not radiate as a whole, but each point or small region sends an independent image into the surrounding medium. Al-Kindī thus made explicit a conception that would be fundamental to subsequent theories of vision.

Al-Kindī was concerned solely with the process of radiation and did not, therefore, incorporate his principle of incoherent punctiform radiation into his own extramission theory of vision. It was Ibn al-Haytham, writing a century and a half later, who showed how one could build a successful intromission theory of vision on al-Kindī's principle. Ibn al-Haytham understood that if each point in the visual field radiated independently in all directions, then each point of the eye would receive radiation from every point in the visual field; the mingling at every point in the eye of rays from different points in the visual field should result in total confusion. To account for clear vision, one had to devise a method whereby each point in the eye would be stimulated by a single point in the visual field—a method, moreover, whereby the points in the eye would share the configuration of the points in the visual field by which they were stimulated.[17]

Ibn al-Haytham solved this problem through the principles of refraction. Pointing out that only one of the multitude of rays emanating from a point in the visual field fell perpendicularly on the surface of the eye, thus entering without refraction, he argued that that ray alone was instrumental in producing visual perception; all other rays were weakened by refraction and rendered ineffective. Moreover, the collection of perpendicular rays constitutes a visual cone, with apex at the centre of the eye and base on the various objects that make up the visual field. Thus was the visual cone of the Euclidean mathematical tradition incorporated, for the first time, into an intromission theory of vision; and thus were all of the mathematical advantages associated with the visual cone (a complete theory of visual perspective) combined, for the first time, with the physical or causal explanations traditionally supplied by intromission theories. In addition to this triumph, Ibn al-Haytham managed to integrate into his theory the anatomical and physiological achievement of Galen and the medical tradition, producing thereby a theory of vision addressed simultaneously to mathematical, physical and medical concerns.

Before the translations of the twelfth and thirteenth centuries, one form or another of the extramission theory dominated Western speculation on vision, owing probably to Platonic and Stoic influence. Augustine declared in his *De Genesi ad litteram* that the light issuing from the eye is fire, produced in the liver and passing from there to the brain and from the brain through 'slender ducts' to the eyes. This light falls on visible objects and reveals them to the sense of sight.

> The emission of rays from our eyes [Augustine wrote] is surely the emission of a certain light. It can be drawn in when we observe that which is near our eyes, thrust out when we observe things along the same line, but far away. When it is drawn in, it does not altogether fail to discern distant things, though it discerns them more obscurely than when it is projected to a distance. Nevertheless, this light which is in the eye is shown to be so scanty that we would see nothing without the assistance of exterior light.[18]

Isidore of Seville, in the seventh century, argued that 'the eyes are also lights *[lumina]*. And they are called lights because light *[lumen]* flows from them, whether from the beginning they contain an enclosed light *[lucem]* or reissue externally a received light in order to produce vision (*Isidori Hispalensis,* XI.1.36–7).

The growing prominence of Plato's *Timaeus* in the twelfth century reinforced the theory of visual fire. In the *Timaeus,* Plato had argued that visual fire emanates from the eye and coalesces with daylight to form 'a single homogeneous body' stretching from the eye to the visible object; this body functions as a medium by which motions are transmitted from the visible object to the soul. Twelfth-century scholars, such as Adelard of Bath and William of Conches, quickly assimilated and elaborated on this Platonic account, raising certain critical questions, but in general reinforcing the conviction that sight is the result of a fiery emanation from the eye (Lindberg 1976:5–6, 91–4).

The relative unanimity of the early Middle Ages on the question of visual theory was quickly dissolved by the translations, which brought to the West the full range of Greek and Arabic thought on the subject. The extramission theory now acquired additional support from Euclid, Ptolemy, al-Kindī and the Galenists—though a close look revealed that these authors had diverged widely on many of the fine points. At the same time, intromission theories appeared, backed by powerful authorities and supported by persuasive arguments. The challenge confronting Western scholars was to select and mediate among the alternatives.

A modest initial effort at resolving the perplexity was undertaken by Grosseteste, who had at least limited familiarity with the intromission theory, which he seems to have been determined to take seriously, while retaining an allegiance to the Platonic theory of visual fire.[19] Grosseteste concludes that there is truth in both opinions. He defends the extramission theory against 'those who consider the part and not the whole', arguing that 'the emission of visual rays' is not 'imagined and without reality'.[20] On the other hand, he clearly believes that the intromission theory is incomplete rather than incorrect; vision, he argues, 'is not completed simply by reception of the sensible form without matter, but in the aforementioned reception and the emergence of radiation from the eye'.[21]

A far more extensive analysis of visual theory was undertaken in the next generation by Albert the Great (d. 1280). In a variety of works, Albert defended Aristotle's intromission theory against all rivals—particularly the intromission theory of the atomists and the extramission theories of Plato, Euclid and al-Kindī. Nonetheless, Albert did not shrink from enlarging Aristotle's theory through the incorporation of pieces of geometrical optics, drawn from Ibn Sīnā, Ibn Rushd and Ibn al-Haytham, and anatomical knowledge derived from the Galenic tradition.[22]

The Western response which, in the long run, would prove most influential was that of Roger Bacon, Albert's contemporary. Bacon was the first Christian scholar to acquire a full mastery of the optical system of Ibn al-Haytham; we do not know exactly when or how he gained access to Ibn al-Haytham's *De aspectibus,* but by the time he began to compose his major optical works in the 1250s or 1260s the doctrines of Ibn al-Haytham had powerfully shaped his understanding of the science of optics. Bacon thus accepted a broad conception of the aims of optics, acknowledging that it was properly concerned with mathematical, physical, anatomical, physiological and even psychological matters.

The essentials of Bacon's theory of vision are all drawn from Ibn al-Haytham. Forms or species emanate in all directions from each point in the visual field. Radiation falling obliquely on an observer's eye is refracted and weakened. Rays incident perpendicularly, the only visually effective ones, form a visual cone that accounts for the mathematical features of visual perception. The physics of perception also receives Bacon's closest attention, worked out in his doctrine of the multiplication of species. Inside the eye, the species are perceived in the crystalline lens and subsequently transmitted through the optical pathway, as Galen and Ḥunayn had defined it, to the brain.[23]

But Bacon had strong syncretistic inclinations. He found Ibn al-Haytham persuasive, but he was not prepared to renounce the authority of Plato, Euclid, Aristotle, Ptolemy, Augustine or al-Kindī. Bacon therefore set out to demonstrate the compatibility of all the major optical authorities; some may have had partial knowledge, but none had been wrong. This raised interesting questions, such as whether Aristotle's transformation of the medium, Ibn al-Haytham's forms and Grosseteste's species were the same thing (Bacon argued that they were). A more difficult problem was to reconcile the intromissionism of Aristotle and Ibn al-Haytham with the extramissionism of Euclid, Ptolemy, Augustine and al-Kindī. Bacon ingeniously resolved this problem by arguing that although Aristotle and Ibn al-Haytham were correct in arguing that intromitted rays were the immediate cause of vision, nothing in their works ruled out a simultaneous radiation of species from the eye—species that serve to ennoble the incoming light or species, preparing them to act on the eye and the visual power.

The details of Bacon's theory need not concern us. What is important is that Bacon presented an imposing synthesis of Greek and Arabic optical knowledge, which would prove influential for more than 300 years. Not only did manuscript copies of Bacon's optical works circulate widely, but Baconian ideas were also widely disseminated through the popular textbooks of Bacon's younger contemporaries, Witelo and John Pecham. At the same time, Ibn al-Haytham's own work continued to circulate and directly shape optical knowledge. The tradition of *perspectiva,* embodying the achievement of Ibn al-Haytham and other Greek and Arabic authors, persisted throughout the fourteenth, fifteenth and sixteenth centuries. When Johannes Kepler confronted the problem of vision at the beginning of the seventeenth century, he began where Ibn al-Haytham had left off.[24]

NOTES

1 Pliny, *Natural History,* II. 8, on shadows. For Pliny's discussion of the eye, see *Natural History*. XI.53–55. There is no adequate account of early Western optical thought. For a brief sketch, see Lindberg (1976:87–90).

2 For a summary of the translation of optical treatises, including citations of the specialized literature on the subject, see Lindberg (1976:209–13).

3 On the translations of Euclid's *Optica,* see Theisen (1979). Three medieval versions of the Euclidean *Catoptrica* have recently been edited by Kenichi Takahashi (1986). Ptolemy's authorship of the *Optica* traditionally attributed to him has recently been questioned by Knorr (1985); for our purposes, the identity of the author is unimportant, and, without meaning to question the cogency of Knorr's argument, I shall continue to refer to this as the *Optica* or *De aspectibus* of Ptolemy.

4 On Euclid's optics see Lejeune (1948).

5 Albert the Great is a good example; see Lindberg (1987b:256–7).

6 McEvoy (1983:631–5): on Grosseteste's optics see Eastwood (1967) and Lindberg (1976:94–102).

7 *De multiplication specierum,* II.2, in *Philosophy of Nature,* pp. 97–105; *The Opus Majus of Roger Bacon* I, 111–17. The particular animated medium Bacon has in mind is the visual spirit that fills the optic nerves. On Bacon and the propagation of light, also see Lindberg (1985).

8 *De multiplication specierum,* II.6, in *Philosophy of Nature* pp. 137–47.

9 *Ibid.,* II.3, pp. 105–11.

10 *Ibid.,* II.4, pp. 117–19; II.7, pp. 147–55.

11 On Witelo, see the recent editions, accompanied by English translation, of Books I

and V of Witelo's *Perspectiva,* by Sabetai Unguru and A.Mark Smith, respectively (*Studio Copernicana,* vols XV and XXIII, Wroclaw: Ossolineum, 1977 and 1983). For a summary account, see David C.Lindberg's 'Witelo' in *Dictionary of Scientific Biography,* XIV, pp. 457–62. On Pecham, see Lindberg (1970).

12 This is not to suggest that the geometrization of optical phenomena is altogether without philosophical implications, but simply to note that the classical rules of geometrical optics have proved their ability to coexist with virtually any theory of the nature of light and to adapt to many different metaphysical environments. To put it quite simply, corporealists and incorporealists subscribe to the same laws of reflection and refraction. See Lindberg (1987a).

13 Sabra, 'Ibn al-Haytham', *in Dictionary of Scientific Biography,* VI, 190–2; see also Rashed (1970a:273).

14 Some historians prefer to characterize Aristotle's theory as a 'mediumistic' or 'modification' theory and set this in opposition to intromission theories. My own preference is to see it as an intromission version of the modification theory.

15 A point stressed in A.Mark Smith (1989).

16 This argument is more fully developed in Lindberg (1976:57–60; 1978a; 339–42).

17 On Ibn al-Haytham's theory of vision, see Lindberg (1976: ch. 4; 1978a:345–9).

18 *De Genesi ad litteram,* I.16.31, translated from Joseph Zycha's edition in *Corpus Scriptorum Ecclesiasticorum Latinorum,* vol. 28, pt. 1 1894, Vienna, p. 23; for an alternative translation see *The Literal Meaning of Genesis,* vol. 1, pp. 37–8.

19 On Grosseteste's theory of vision see Lindberg (1976:100–1). Grosseteste's task was complicated by the fact that he had at his disposal Michael Scot's translation of Aristotle's *De animalibus,* where, owing to mistranslation. Aristotle appeared to defend the extramission theory; see Wingate (1931:78).

20 *De iride,* quoted from Grant (1974:389).

21 *Commentarius,* p. 386: this passage was first noted and translated by Crombie (1953:114).

22 Lindberg (1976:104–6:1987b).

23 On Bacon's theory of vision, see Lindberg (1976:107–16).

24 On the influence of Arabic optics, see Lindberg's 'Introduction' to the reprint of Ibn al-Haytham, *Opticae Thesaurus,* pp. xxi–xxv; Lindberg (1976: ch 6–9).

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